

# Repulsive Knot Energies and Pseudodifferential Calculus for O’Hara’s Knot Energy Family $E^{(\alpha)}$ , $\alpha \in [2, 3)$

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Received XXXX, revised XXXX, accepted XXXX  
Published online XXXX

**Key words** Knot energy, bilinear Fourier multiplier, Möbius Energy, regularity  
**MSC (2000)** 42A45, 53A04, 57M25

We develop a precise analysis of J. O’HARA’s knot functionals  $E^{(\alpha)}$ ,  $\alpha \in [2, 3)$  that serve as self-repulsive potentials on (knotted) closed curves. First we derive continuity of  $E^{(\alpha)}$  on injective and regular  $H^2$  curves and then we establish Fréchet differentiability of  $E^{(\alpha)}$  and state several first variation formulae. Motivated by ideas of Z.-X. HE in his work on the specific functional  $E^{(2)}$ , the so-called Möbius energy, we prove  $C^\infty$ -smoothness of critical points of the appropriately rescaled functionals  $\tilde{E}^{(\alpha)} = \text{length}^{\alpha-2} E^{(\alpha)}$  by means of fractional Sobolev spaces on a periodic interval and bilinear Fourier multipliers.

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## Introduction

*Geometric knot theory* is a recent subfield of knot theory which started from the investigation of *knot energies* that, in contrast to classical knot theory [10, 17], measure geometric and analytic properties of a given knot rather than its topological knot class. The development of this new area began in 1988 with an exposé of FUKUHARA [20] dealing with polygons. Subsequently O’HARA [34] proposed the first knot energy defined on smooth curves.

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The general idea is to search for representatives possessing a particularly “nice” shape within a given knot class. Besides requirements on its smoothness, such a knot is expected to look as little entangled as possible, i. e. different strands of this representative are wide apart, having preferably large distances. In order to achieve the latter, one needs to model *self-avoidance* phenomena, i. e. the energy blows up on sequences of embedded curves converging to a curve with a *self-intersection*. In fact, this property essentially characterizes a knot energy. By imposing self-avoidance one hopes not to run into the danger of leaving the ambient knot class, e. g. while following a solution of a gradient flow. Unfortunately, small knots may pull tight in limiting processes which is *not* prevented by this definition.

Besides the aim to distinguish between distinct knot classes, the investigation of knot energies may also have its impact on the sciences, e. g. bio-chemistry and theoretical physics, whenever repulsive forces of fibres are modeled. Possible applications are the deformation of a thin fibre charged with electrons lying in a viscous liquid [20], the behaviour of protein foldings [32], or the motion of knotted DNA structures in electrophoresis gels [12]. The evolution of these objects is obtained by studying the negative gradient flow associated to the respective knot energy. On the other hand, attraction phenomena may also be modeled by a corresponding positive gradient flow; see [2] for an example from mathematical biology discussing interaction between pairs of filaments via cross-linkers.

In our terminology, a *knot* denotes a closed embedded curve  $\gamma \in AC(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d)$ , i. e.  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$  being absolutely continuous and  $\ell$ -periodic. Additionally, we restrict to *regular* curves which satisfy  $\dot{\gamma} \neq 0$  a. e. In 1992, O’HARA defined the family of  $(\alpha, p)$ -knot functionals [35],  $\alpha, p \in (0, \infty)$ ,

$$E^{\alpha,p}(\gamma) := \iint_{(\mathbb{R}/\ell\mathbb{Z})^2} \left( \frac{1}{|\gamma(s) - \gamma(t)|^\alpha} - \frac{1}{D_\gamma(s,t)^\alpha} \right)^p |\dot{\gamma}(s)| |\dot{\gamma}(t)| \, ds \, dt,$$

where  $\mathcal{L}(\gamma) := \int_0^\ell |\dot{\gamma}(\theta)| \, d\theta$  is the length of  $\gamma$  and  $D_\gamma(s,t) := \min(\mathcal{L}(\gamma|_{[s,t]}), \mathcal{L}(\gamma) - \mathcal{L}(\gamma|_{[s,t]}))$  denotes the (intrinsic) distance of  $\gamma(s)$  and  $\gamma(t)$  on the curve  $\gamma$  provided  $s \in [t - \frac{\ell}{2}, t + \frac{\ell}{2}]$ . For general  $s$ , the function  $(s,t) \mapsto D_\gamma(s,t)$  is extended periodically to  $(\mathbb{R}/\ell\mathbb{Z})^2$  which is homeomorphic to the torus. The factor  $|\dot{\gamma}(s)| |\dot{\gamma}(t)|$  guarantees invariance under reparametrization. The functionals are knot energies (i. e. they are bounded below and possess the self-avoidance property mentioned above) iff  $\alpha p \geq 2$  and well-defined (i. e.  $E^{\alpha,p} \neq \infty$ ) iff  $(\alpha - 2)p < 1$ , see [36, Thm. 1.1] and [1, Proof of Cor. 3], which we will assume from now on. So the functionals  $E^{(\alpha)} := E^{\alpha,1}$  together with the corresponding rescaled analoga  $\tilde{E}^{(\alpha)} := \mathcal{L}^{\alpha-2} E^{(\alpha)}$  are relevant for  $\alpha \in [2, 3)$  only. The singularity in the integrand of  $E^{\alpha,p}$ , occurring as  $|\gamma(s) - \gamma(t)| \rightarrow 0$ , penalizes pairs of points  $(\gamma(s), \gamma(t))$  that have small Euclidean but large (intrinsic) distance on the curve. In this situation, the contribution of the minuend  $|\gamma(s) - \gamma(t)|^{-\alpha}$  is rather high and cannot be absorbed by the subtrahend  $D_\gamma(s,t)^{-\alpha}$ . Curves with a self-intersection produce an unrecoverable singularity which leads to infinite energy.

The main results of this paper concern regularity properties of  $E^{(\alpha)}$ . The first goal is to rigorously establish the Fréchet differentiability. In the following statements,  $H^\alpha(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  denotes the Sobolev space of (fractional) order  $\alpha \in [2, 3)$  consisting of closed curves in the  $d$ -dimensional Euclidean space, see (1.3).

**Theorem 1.21** *Let  $\gamma, h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  with  $\gamma$  being injective and  $|\dot{\gamma}| > 0$ ,  $\alpha \in [2, 3)$ . Then  $E^{(\alpha)}$  is Fréchet differentiable at  $\gamma$  in direction of  $h$ . If  $\gamma$  is parametrized by arc-length, its first variation is given by*

$$I^{(\alpha)}(\gamma, h) = \lim_{\varepsilon \searrow 0} \iint_{W_\varepsilon} \left\{ (\alpha - 2) \frac{\langle \dot{\gamma}(t), \dot{h}(t) \rangle}{|w|^\alpha} + 2 \frac{\langle \dot{\gamma}(t), \dot{h}(t) \rangle}{|\Delta\gamma|^\alpha} - \alpha \frac{\langle \Delta\gamma, \Delta h \rangle}{|\Delta\gamma|^{\alpha+2}} \right\} \, dw \, dt, \quad (1.20)$$

otherwise, one has to use a reparametrization formula, cf. (1.25).

This statement is used to consider the Euler-Lagrange equation which finally reveals

**Theorem 2.23** *Let  $\gamma \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $\alpha \in [2, 3)$ , be an injective curve, parametrized by arc-length with cube-integrable curvature  $|\ddot{\gamma}|$ . If  $\gamma$  is a critical point of the rescaled functional  $\tilde{E}^{(\alpha)}$  then it is  $C^\infty$ -smooth.*

The most prominent case is  $E := E^{(2)} = E^{2,1}$  which was extensively studied by FREEDMAN, HE, and WANG, who coined the name *Möbius Energy* for  $E$  because of its invariance under Möbius transformations in

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**Note added in proof.** In a forthcoming paper [9], we are able to close the gap between the initial regularity of finite-energy curves [5] and the regularity requirement in Theorem 2.23, i. e. we show that critical points of  $\tilde{E}^{(\alpha)}$  in  $H^{(\alpha+1)/2}$  parametrized by arc-length are  $C^\infty$ -smooth.

$\mathbb{R}^3$ . In their seminal paper [19] they prove the existence of minimizers in prime knot classes and  $C^{1,1}$  regularity of local minimizers in arbitrary knot classes (which is improved to  $C^\infty$  regularity by Theorem 2.23 of this text). Furthermore they also state a formula for the first variation. Unfortunately, they just claim  $\delta E(\gamma, h) = \lim_{\varepsilon \searrow 0} \iint_{|s-t| \geq \varepsilon} \left( \frac{d}{d\tau} \Big|_{\tau=0} \frac{|\dot{\gamma}_\tau(s)||\dot{\gamma}_\tau(t)|}{|\dot{\gamma}_\tau(s) - \dot{\gamma}_\tau(t)|} \right) ds dt$ , where  $\gamma_\tau := \gamma + \tau h$ , and check that  $\frac{d}{d\tau} \Big|_{\tau=0} \frac{|\dot{\gamma}_\tau(s)||\dot{\gamma}_\tau(t)|}{|\dot{\gamma}_\tau(s) - \dot{\gamma}_\tau(t)|} = \mathcal{O}(1)$  as  $|s-t| \rightarrow 0$  which does *not* prove their assertion. Their conclusion that  $E$  is Fréchet differentiable is incorrect, too; a proof for this statement is presented in this text. Further details on O'HARA's energy functionals may be found in [1, 8, 35, 37]. In the more complicated situation of higher-dimensional submanifolds hardly anything is known analytically, see [4, 32].

A couple of years after the joint paper [19], HE published his inspiring investigation on the Euler-Lagrange equation and the heat flow associated to the Möbius Energy [30]. As to smoothness of critical points, he treats the special case  $\alpha = 2$  of Theorem 2.23 above. Unfortunately, HE's arguments fail to work correctly which will be commented on in detail below. However, we will furnish a rigorous proof in the more general situation  $\alpha \in [2, 3)$  using some of HE's brilliant ideas. For a rigorous analysis of the gradient flow we refer to BLATT [6, 7].

Yet another famous example of a knot energy is the reciprocal of *thickness*  $\Delta[\cdot]$  which can be characterized by means of the *global radius of curvature*  $\varrho_G$  defined by GONZALEZ and MADDOCKS [24]. This leads to the concept of *ideal knots*. Existence theory is discussed in [13, 22, 23, 25]. Regularity theory regarding ideal knots turns out to be rather involved, see [11, 42, 43]. In fact, an explicit analytical characterization of the shape of a (non-trivial) ideal knot has not been found yet, so the state of the art is numerical visualization, cf. [3, 14, 15, 21, 26, 44]. There are many generalizations of this concept, e. g. higher-dimensional analoga, see [45, 46, 47, 48, 49, 50]. We also refer to [37] for a detailed outline on several knot energies and the respective properties.

Before describing the structure of the present paper let us outline HE's regularity approach [30] for the Möbius energy  $E^{(2)}$ , which motivated our investigations on the general potentials  $E^{(\alpha)}$ .

## 0.1 HE's approach for the Möbius Energy

**Derivation of the derivative.** HE's ingenious idea is to introduce a quadratic functional  $Q$ , appearing as  $Q^{(2)}$  in the present text, which turns out to be a "linearization" of the first variation of  $E$ . Without a further comment, HE cites the existence of the Gâteaux derivative of  $E$  just from [19]. Moreover, HE does neither prove nor cite a proof for the "elementary" fact that arc-length reparametrization preserves  $H^2$  convergence.

The reasoning for the assertion that  $E$  admits Gâteaux derivatives up to order  $k+1$  when restricted to  $H^{k+2}$  for all  $k \in \mathbb{N}$  is questionable [30, Lemma 4.5]. HE does not explain why repeated interchanging of differentiation, integration and a limit process  $\lim_{\tau \searrow 0} \lim_{\varepsilon \searrow 0} \iint_{|s-t| \geq \varepsilon} = \lim_{\varepsilon \searrow 0} \iint_{|s-t| \geq \varepsilon} \lim_{\tau \searrow 0}$  should be a valid operation. On the other hand, he provides a nice geometric argument for the fact that  $\delta E(\gamma, h) = 0$  for variational fields  $h$  tangent to  $\gamma$  [30, Lemma 4.6].

**Critical points and the bootstrapping.** HE carries out a clever transformation of the weak Euler-Lagrange equation of  $E$ . However, suitable background for the underlying Sobolev theory is not provided, so HE can neither properly initialize nor carry out the bootstrapping. This accumulates at the commutator estimate [30, Lemma 5.2]. HE claims  $\|P_{f^\perp} \Delta^{1/2} g - \Delta^{1/2} P_{f^\perp} g\|_{H^\sigma} \leq C_\sigma \|f\|_{H^{1+\sigma}} \|g\|_{H^{1/2+\sigma}}$  for  $\sigma > 0$ , where  $P_{f^\perp} h := h - \langle h, f \rangle_{\mathbb{R}^d} f$ . The norm  $\|g\|_{H^{1/2+\sigma}}$  is not sufficient in general. For small  $\sigma \geq 0$  one needs at least  $\|g\|_{H^{1/2+\sigma+\varepsilon}}$ ,  $\varepsilon > 0$ , as  $H^s \hookrightarrow L^\infty$  for  $s > \frac{1}{2}$  only. However, in the first step of the bootstrapping process, HE uses the estimate for  $\sigma = 0$  and  $\varepsilon = 0$  in order to show  $\Delta^{-1/4} N_\gamma \dot{\gamma} \in (L^2)^*$ . For a proof of the commutator estimate, he refers to KATO and PONCE [31]. It is not obvious how to carry over their result. They treat the situation  $\|\Delta^{\sigma/2}(fg) - f\Delta^{\sigma/2}g\|_{L^2}$  (with  $\text{Id} - \Delta$  instead of  $\Delta$ ) while HE essentially needs an estimate for  $\|\Delta^{(\sigma+1)/2}(fg) - \Delta^{\sigma/2}(f\Delta^{1/2}g)\|_{L^2}$  together with a statement on the product of Sobolev functions.

Moreover, due to the lack of a suitable theory, HE cannot provide results on higher regularity of the "monster" operator  $M$  which he shows to belong to  $L^1$  only. This adds to the fact that the bootstrapping does not reach beyond the first step.

## 0.2 Exposé of the present work

**Section 1. Fréchet differentiability.** We start with a brief introduction concerning Fourier theory and fractional Sobolev spaces  $H^s$ . Assuming arc-length parametrization, we begin with the investigation of  $E^{(\alpha)}$ , whose “linearization” is given by the bilinear form  $Q^{(\alpha)}$ , see (1.6). It turns out that  $E^{(\alpha)}(\gamma)$  may be written as  $\frac{\alpha}{2}Q^{(\alpha)}(\gamma, \gamma) +$  “lower order terms”, cf. (1.8). This bilinear form defined on  $H^\sigma \times H^{\alpha+1-\sigma}$  is discussed in Proposition 1.3.

The functional  $Q^{(\alpha)}$  is quite remarkable as it admits to characterize the Sobolev spaces  $H^s$ ,  $s \in (1, 2)$ , without the aid of Fourier series. More precisely,  $f \in H^{1+\beta} \iff Q^{(1+2\beta)}(f, f) < \infty$  for  $\beta \in (0, 1)$ , and  $f \mapsto \sqrt{Q^{(1+2\beta)}(f, f)}$  is a seminorm on  $H^{1+\beta}$ . The corresponding functional for  $H^{0+\beta}$  is given by  $f \mapsto \iint_{(\mathbb{R}/2\pi\mathbb{Z})^2} \frac{|f(s)-f(t)|^2}{|s-t|^{1+2\beta}} ds dt$ . Similar norms for  $H^{k+\beta}$ ,  $k \in \mathbb{N}$ ,  $\beta \in (0, 1)$ , may be defined inductively, see HE’s note [29] for the  $\mathbb{R}^n$  case which can be transferred to the periodic setting using the technique from the proof of Proposition 1.3. A similar characterization that does not involve Fourier series is furnished by seminorms using difference operators, see [28, Exerc. 6.3.9] for the situation in  $\mathbb{R}^n$ .

The “lower order terms” mentioned above are treated in Corollary 1.5 using ideas from [30, Lemma 4.3]. We finally obtain continuity of  $E^{(\alpha)}$  on injective embedded  $H^2$  curves with arbitrary parametrization (assuming that the first derivative never vanishes) by Theorem A.1 from the appendix.

It is rather simple to formally differentiate the integrand of  $E^{(\alpha)}(\gamma + \tau h)$  outside an  $\varepsilon$ -neighborhood of the diagonal  $s = t$  with respect to  $\tau$  at  $\tau = 0$ . But it is not clear at all why interchanging of differentiation with respect to  $\tau$ , integration, and a limit process  $\varepsilon \searrow 0$  should be an admissible operation. The main problem consists in the fact that, since the integrand of  $E^{(\alpha)}(\gamma + \tau h)$  is singular, its derivative is even more singular and fails to be  $L^1(\mathbb{R}/2\pi\mathbb{Z} \times (-\pi, \pi))$ . So we may not immediately apply LEBESGUE’s theorem on dominated convergence. At least the principal value exists and is finite.

The approximate functional  $E_\varepsilon^{(\alpha)}$ , which is obtained from  $E^{(\alpha)}$  by removing the “singular strip”  $\{|s-t| < \varepsilon\}$  from the integration domain, can be differentiated just via LEBESGUE’s theorem. Its first variation  $I_\varepsilon^{(\alpha)}$  turns out to be a continuous mapping  $\tau \mapsto I_\varepsilon^{(\alpha)}(\gamma + \tau h, h)$  for  $|\tau| \ll 1$  and  $\varepsilon > 0$ . If  $\alpha > 2$  we also have to consider the first variation of  $D_\gamma$ , cf. Lemma 1.13. Interestingly, the functional  $I_\varepsilon^{(\alpha)}$  does not involve  $D_\gamma$  nor its derivative which is compatible with [19, Lemma 6.1]. The technical part is now to show that  $\tau \mapsto \lim_{\varepsilon \searrow 0} I_\varepsilon^{(\alpha)}(\gamma + \tau h, h)$  is in fact continuous for  $|\tau| \ll 1$ , see Lemma 1.14. Finally, Lemma 1.15 reveals the existence of the first variation  $I^{(\alpha)}$ .

Using ideas from [30, Lemma 4.2] we see that  $I^{(\alpha)}$  indeed continuously extends to  $\gamma, h \in H^2$  where  $\gamma$  is at first parametrized by arc-length, see Lemma 1.17, and then transfer this result to arbitrary regular  $H^2$  curves via reparametrization to arc-length (1.25), cf. Corollary 1.16. Approximating  $\gamma, h \in H^2$  by smooth functions and using Lemma 1.15, we finally obtain Fréchet differentiability in Theorem 1.21.

In the last paragraph of the first section, we state some formulae for the first variation that no longer involve derivatives of  $D_\gamma$ , see (1.26), (1.27). The first derivative of  $E^{(\alpha)}$  given in (1.28) admits a negative gradient flow, see Remark 1.25.

**Section 2. Smoothness of critical points.** In the present work we study *bilinear Fourier multipliers* in order to investigate the regularity of products  $f \cdot g$  and commutators  $[J, f]g := J(fg) - f(Jg)$  for given  $H^s$  functions  $f$  and  $g$ , where  $J$  denotes the “standard” differential operator on  $H^s$ . Provided  $s > \frac{1}{2}$ , the multiplier  $\mathcal{M}_\mu(f, g)$  defined in (2.1) belongs to  $H^s$ . In spite of the lack of a “Leibniz rule” for  $J$  and its (non-even) powers, we obtain at least an  $L^2$  estimate for  $J^s(fg)$  in terms of the  $L^2$  norms of  $J^s f$  and  $J^s g$  (provided  $s > \frac{1}{2}$ ). A more general statement on multiplication in fractional Sobolev spaces can be found in [38, Thm. 9.5]. We also obtain weaker results for  $s \leq \frac{1}{2}$ . Note that the symbol  $\mu_{k,l}$  of  $\mathcal{M}_\mu$  is uniformly bounded and does not depend on the variable  $t$ . The contrary situation of non-constant symbols  $(\mu_{k,l})_{k,l \in \mathbb{Z}} \in (H^s(\mathbb{R}/2\pi\mathbb{Z}))^{\mathbb{Z}^2}$  becomes rather involved, see e. g. [16, Thm. 12 (p. 55), Prop. 2 (p. 154)] or [33, Chap. 13], that treat the situation in  $\mathbb{R}^n$ .

By the scaling property, we may expect critical points for the rescaled functionals  $\tilde{E}^{(\alpha)}$  only or, equivalently, have to impose a side condition on the length of the curve. As in [30], we have to restrict to test functions in the orthogonal complement of  $\dot{\gamma}$  in order to derive a weak Euler-Lagrange equation (2.11) that allows to separate the highest-order terms. The tedious “remainder term”  $M^{(\alpha)}$  which essentially forms a product of derivatives of  $\gamma$  with shifted arguments is thoroughly treated in Lemmata 2.20 and 2.21, filling one of HE’s major gaps. The proof

of Theorem 2.23 finally contains the bootstrapping argument. Due to the limited regularity, one can only proceed in steps  $H^s \Rightarrow H^{s+\frac{1}{2}-\varepsilon}$  at the beginning.

It is not clear whether similar properties apply for the energies  $E^{\alpha,p}$  where  $p \neq 1$ . In this case we do not obtain an easy decomposition of  $E^{\alpha,p}$  as in (1.8). The computation (1.21) by which we dispose the derivative of  $D_\gamma$  does not hold, but we face another factor  $\left(\frac{1}{|\gamma(s)-\gamma(t)|^\alpha} - \frac{1}{D_\gamma(s,t)^\alpha}\right)^{p-1}$  in the integrand.

For more detailed proofs of the statements in this text we refer to [40].

## 1 Fréchet differentiability

We briefly recall some basics of Fourier series. For the general theory of Fourier analysis we refer to, e. g., [27, Ch. 3] and [41, 4.26], details regarding Sobolev spaces may be found in [18, Ch. 5], [28, Sect. 6.2], and [51, Sect. 4.1 – 4.3]. Let  $d \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ ,  $m \in \mathbb{N}$ , and  $s \in \mathbb{R}$ .

We start with the definition of a rescaled  $L^2$  scalar product

$$\langle f, g \rangle_{L^2} := \int_0^{2\pi} \langle f(t), g(t) \rangle_{\mathbb{R}^d} dt := \frac{1}{2\pi} \int_0^{2\pi} \langle f(t), g(t) \rangle_{\mathbb{R}^d} dt = \frac{1}{2\pi} (f, g)_{L^2}$$

which induces the norm  $\|f\|_{L^2} := \sqrt{\langle f, f \rangle_{L^2}} = \frac{1}{\sqrt{2\pi}} \|f\|_{L^2}$ .

Defining the  $L^2$  basis functions  $\phi_k : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C}$ ,  $t \mapsto e^{ikt}$ ,  $k \in \mathbb{Z}$ , any  $L^2$  function  $f : (0, 2\pi) \rightarrow \mathbb{R}^d$  may be written in terms of its *Fourier coefficients*  $\hat{f}_k := \int_0^{2\pi} f(t) \phi_{-k}(t) dt$  via  $f = \sum_{k \in \mathbb{Z}} \hat{f}_k \phi_k$ , which converges in  $L^2$ . Vice versa, any  $\ell^2$  sequence  $(\hat{f}_k)_{k \in \mathbb{Z}} \subset \mathbb{C}^d$  satisfying  $\hat{f}_k = \hat{f}_{-k}$  for all  $k \in \mathbb{Z}$  defines a function in  $L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  by  $\sum_{k \in \mathbb{Z}} \hat{f}_k \phi_k$ . This is precisely the isometry between the two Hilbert spaces  $L^2$  and  $\ell^2$  arising from the RIESZ-FISCHER theorem.

By PARSEVAL's theorem, for all  $f, g \in L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,

$$\langle f, g \rangle_{L^2} = \left\langle \left( \hat{f}_l \right)_{l \in \mathbb{Z}}, \left( \hat{g}_l \right)_{l \in \mathbb{Z}} \right\rangle_{\ell^2} := \sum_{k \in \mathbb{Z}} \langle \hat{f}_k, \hat{g}_k \rangle_{\mathbb{C}^d}. \quad (1.1)$$

These facts are used for the following

**Lemma 1.1** *Let  $f \in H^m(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $m \in \mathbb{N}$ . Then the sequence  $(\hat{f}_k)_{k \in \mathbb{Z}} \subset \mathbb{C}^d$  defined by  $\hat{f}_k := \int_0^{2\pi} f(t) \phi_{-k}(t) dt$  satisfies*

$$\|f\|_{H^m} := \sqrt{\sum_{k \in \mathbb{Z}} (1+k^2)^m |\hat{f}_k|^2} < \infty. \quad (1.2)$$

*On the other hand, any sequence  $(\hat{f}_k)_{k \in \mathbb{Z}}$  satisfying  $\hat{f}_k = \hat{f}_{-k}$  and (1.2) defines a function  $\sum_{k \in \mathbb{Z}} \hat{f}_k \phi_k \in H^m(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . The norms  $\|\cdot\|_{H^m}$  and  $\|\cdot\|_{H^m}$  are equivalent.*

By this property, one can define *fractional order Sobolev spaces*

$$H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \mid \|f\|_{H^s} := \sqrt{\langle f, f \rangle_{H^s}} < \infty \right\}, \quad s \geq 0, \quad (1.3)$$

which are Hilbert spaces equipped with the scalar product

$$\langle f, g \rangle_{H^s} := \sum_{k \in \mathbb{Z}} (1+k^2)^s \langle \hat{f}_k, \hat{g}_k \rangle_{\mathbb{C}^d}.$$

The definition of  $H^s$  induces an isometry  $H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \rightarrow H^{s-\sigma}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $s \geq 0$ ,  $\sigma \in (-\infty, s]$ ,

$$J^\sigma : f \mapsto \sum_{k \in \mathbb{Z}} (1+k^2)^{\sigma/2} \hat{f}_k \phi_k,$$

so  $f \in H^s \iff J^\sigma f \in H^{s-\sigma} \iff J^s f \in L^2$ . Note that  $J^2 = \text{id} - \frac{d^2}{dt^2}$ . Moreover, given  $s \geq 0, \sigma, \tau \geq -s$ ,  $\varphi \in H^{s+\sigma} \cap H^{s+(\sigma+\tau)/2}, \psi \in H^{s+\tau} \cap H^{s+(\sigma+\tau)/2}$ , we obtain  $\langle J^\sigma \varphi, J^\tau \psi \rangle_{H^s} = \langle \varphi, \psi \rangle_{H^{s+(\sigma+\tau)/2}}$ , which can be used in many ways, e. g.  $\langle J^\sigma f, g \rangle_{L^2} = \langle f, J^\sigma g \rangle_{L^2} = \langle f, g \rangle_{H^{\sigma/2}}, \sigma \geq 0$ .

We denote by  $C_{\text{ir}}^m$  and  $H_{\text{ir}}^s$  ( $m \in \mathbb{N}, s \geq 2$ ) functions belonging to  $C^m$  or  $H^s$  respectively which are additionally injective and regular. In our situation *injectivity* is equivalent to embeddedness of the curves and *regularity* means that the first derivative never vanishes, i. e.  $|\dot{\gamma}(\cdot)| \geq c > 0$ . By  $C_{\text{ia}}^m$  and  $H_{\text{ia}}^s$  ( $m \in \mathbb{N}, s \geq 2$ ) we will denote curves that are also parametrized by arc-length, i. e.  $|\dot{\gamma}(\cdot)| \equiv 1$ .

### 1.1 Continuity of $E^{(\alpha)}$ on $H_{\text{ir}}^2$

In this section we prove the (sequential) continuity of  $E^{(\alpha)}$  on  $H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ , i. e. we restrict ourselves to curves parametrized by arc-length. To this end we will, following [30, Lemma 4.4], decompose  $E^{(\alpha)}(\gamma)$  for a curve  $\gamma \in H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  introducing and investigating three important functionals,  $Q^{(\alpha)}$ ,  $X$ , and  $\Omega^{(\alpha)}$ . Using a reparametrization theorem, to which the next section is devoted, we may transfer this result to the larger set  $H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .

Let  $W_\varepsilon := [0, 2\pi] \times ([-\pi, -\varepsilon] \cup [\varepsilon, \pi])$  for  $\varepsilon \in [0, \pi]$  and define for any  $f, g, \gamma \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $(t, w) \in W_0$ ,  $\varepsilon \in (0, \pi]$ ,

$$Q_\varepsilon^{(\alpha)}(f, g) := \iint_{W_\varepsilon} \left( \langle \dot{f}(t), \dot{g}(t) \rangle_{\mathbb{R}^d} - \frac{\langle f(t+w) - f(t), g(t+w) - g(t) \rangle}{w^2} \right) \frac{dw dt}{|w|^\alpha},$$

$$X_\gamma(t, w) := 2 \int_0^1 (1-u) \langle \dot{\gamma}(t+uw) - \dot{\gamma}(t), \ddot{\gamma}(t+uw) \rangle du - w \left| \int_0^1 (1-u) \ddot{\gamma}(t+uw) du \right|^2, \quad (1.4)$$

$$\Omega_\gamma^{(\alpha)}(t, w) := \frac{1}{4} \alpha(\alpha+2) \int_0^1 \frac{1-\mu}{(1-\mu w X_\gamma(t, w))^{\alpha/2+2}} d\mu. \quad (1.5)$$

Note that  $\Omega_\gamma^{(2)}(t, w) = (1 - w X_\gamma(t, w))^{-1}$ . Moreover, if existing,

$$Q^{(\alpha)}(f, g) := \lim_{\varepsilon \searrow 0} Q_\varepsilon^{(\alpha)}(f, g). \quad (1.6)$$

Finally, we define the approximative functionals

$$E_\varepsilon^{(\alpha)}(\gamma) := \iint_{W_\varepsilon} \underbrace{\left( \frac{1}{|\gamma(t+w) - \gamma(t)|^\alpha} - \frac{1}{D_\gamma(t, t+w)^\alpha} \right)}_{=: f^{(\alpha)}(\gamma; t, w)} |\dot{\gamma}(t+w)| |\dot{\gamma}(t)| dw dt \quad (1.7)$$

which of course satisfy  $E^{(\alpha)} = \lim_{\varepsilon \searrow 0} E_\varepsilon^{(\alpha)}$ . By  $|\gamma(t+w) - \gamma(t)|^2 \stackrel{(1.4)}{=} w^2 (1 - w X_\gamma(t, w))$ , see [30, (4.8)], we obtain

$$Q_\varepsilon^{(\alpha)}(\gamma, \gamma) = \iint_{W_\varepsilon} w |w|^{-\alpha} X_\gamma(t, w) dw dt \quad \text{for } \gamma \in H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d).$$

Applying the Taylor formula with integral remainder for  $x \mapsto (1+x)^{-\alpha/2}$ ,  $x > -1$ , essentially gives as in [30, Lemma 4.4 (ii)]

**Proposition 1.2** *For any  $\gamma \in H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  we obtain*

$$E_\varepsilon^{(\alpha)}(\gamma) = \frac{\alpha}{2} Q_\varepsilon^{(\alpha)}(\gamma, \gamma) + \iint_{W_\varepsilon} \Omega_\gamma^{(\alpha)}(t, w) |w|^{2-\alpha} X_\gamma(t, w)^2 dw dt. \quad (1.8)$$

In order to prove continuity we have to investigate the functionals involved in (1.8). We start with the first term.

**Proposition 1.3** (i) For each  $\sigma \in [0, \alpha + 1]$ , the functional  $Q^{(\alpha)}$  is bilinear and bounded, hence continuous, on

$$(H^\sigma \times H^{\alpha+1-\sigma})(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d).$$

(ii) There is a linear and bounded (pseudodifferential) operator (defined via a Fourier multiplier)

$$L^{(\alpha)} : H^{s+2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \rightarrow H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \quad \text{for each } s \geq 0$$

and a constant  $a^{(\alpha)} \in \mathbb{R}$  such that, for  $f \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $g \in H^{\alpha-1}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,

$$Q^{(\alpha)}(f, g) = a^{(\alpha)} \langle J^2 f, J^{\alpha-1} g \rangle_{L^2} + \langle L^{(\alpha)} f, g \rangle_{L^2}. \quad (1.9)$$

This proposition generalizes [30, Lemma 2.3] to  $\alpha \in [2, 3)$ .

*Proof.* Linearity in both arguments is obvious. Since any  $L^2$  function is uniquely determined by its Fourier series, we obtain for  $f \in H^\sigma$ ,  $g \in H^{\alpha+1-\sigma}$ ,  $\sigma \in [0, \alpha + 1]$ ,

$$Q^{(\alpha)}(f, g) = 4\pi \int_0^\pi \sum_{k \in \mathbb{Z}} \langle \hat{f}_k, \hat{g}_k \rangle_{\mathbb{C}^d} \frac{F(kw)}{w^{2+\alpha}} dw, \quad (1.10)$$

where  $F(t) := t^2 - 2 + e^{it} + e^{-it} = t^2 - 2 + 2 \cos t$ , which turns out to be even and monotone increasing on  $\{t \geq 0\}$ , for  $\dot{F}(t) = 2(t - \sin t) \geq 0$ . Now  $F(0) = 0$  implies that  $F$  is non-negative on  $\mathbb{R}$ , so

$$\left| Q^{(\alpha)}(f, g) \right| \leq \sum_{k \in \mathbb{Z}} q_k \left| \hat{f}_k \right| \left| \hat{g}_k \right|, \quad \text{where } q_k := 4\pi \int_0^\pi \frac{F(kw)}{w^{2+\alpha}} dw. \quad (1.11)$$

Using integration by parts we can decompose  $q_k = a^{(\alpha)} |k|^{\alpha+1} + b_{|k|}^{(\alpha)} k^2 + c_{|k|}^{(\alpha)}$ , where, for  $k \geq 0$ , the coefficients

$$\begin{aligned} a^{(\alpha)} &:= \frac{8\pi \lambda_\infty^{(\alpha)}}{\alpha(\alpha+1)(\alpha-1)}, & \lambda_\infty^{(\alpha)} &= \lim_{k \rightarrow \infty} \lambda_k^{(\alpha)}, \\ b_k^{(\alpha)} &:= \frac{4\pi}{\alpha+1} \left[ \frac{2}{\alpha(\alpha-1)} \left( (\lambda_k^{(\alpha)} - \lambda_\infty^{(\alpha)}) k^{\alpha-1} \right) - \pi^{1-\alpha} \left( 1 + \frac{2}{\alpha} + 2 \frac{1 - (-1)^k}{\alpha(\alpha-1)} \right) \right], \\ c_k^{(\alpha)} &:= \frac{8}{\alpha+1} \cdot \frac{1 - (-1)^k}{\pi^\alpha} \end{aligned} \quad (1.12)$$

are bounded in  $k$  due to Lemma 1.4 which defines  $\lambda_k^{(\alpha)} = \int_0^{k\pi} \frac{\sin t}{t^{\alpha-1}} dt$ . By  $(1+k^2)^{s/2} - |k|^s = \mathcal{O}((1+k^2)^{s/2-1})$  as  $k \rightarrow \infty$  for  $s > 0$ ,  $k \in \mathbb{Z}$  we may also write

$$q_k = a^{(\alpha)} (1+k^2)^{(\alpha+1)/2} + \mathcal{O}(1+k^2) \quad \text{as } k \rightarrow \infty, \quad (1.13)$$

where the Landau symbol depends on  $\alpha$ . By (1.11), (1.13) and SCHWARZ's inequality we arrive at

$$\left| Q^{(\alpha)}(f, g) \right| \leq \sum_{k \in \mathbb{Z}} \mathcal{O} \left( (1+k^2)^{(\alpha+1)/2} \right) \left| \hat{f}_k \right| \left| \hat{g}_k \right| \leq C_\alpha \|f\|_{H^\sigma} \|g\|_{H^{\alpha+1-\sigma}}$$

with some constant  $C_\alpha$  depending only on  $\alpha$ . This proves (i). In order to show (ii) we first note, that we just have found a majorant for (1.10) which leads to  $Q^{(\alpha)}(f, g) = \sum_{k \in \mathbb{Z}} q_k \langle \hat{f}_k, \hat{g}_k \rangle_{\mathbb{C}^d}$ . Choosing  $\sigma := 2$ , we deduce (1.9) from (1.13) by defining  $L^{(\alpha)} f := \sum_{k \in \mathbb{Z}} \left( a^{(\alpha)} \left( |k|^{\alpha+1} - (1+k^2)^{(\alpha+1)/2} \right) + b_{|k|}^{(\alpha)} k^2 + c_{|k|}^{(\alpha)} \right) \hat{f}_k \phi_k = \sum_{k \in \mathbb{Z}} \mathcal{O}(1) \widehat{J^2 f}_k \phi_k$ .  $\square$

The following statements are proved by elementary means.

**Lemma 1.4** For  $k \in \mathbb{N} \cup \{0\}$  we define

$$\lambda_k^{(\alpha)} := \int_0^{k\pi} \frac{\sin t}{t^{\alpha-1}} dt.$$

These values are well-defined and converge to  $\lambda_\infty^{(\alpha)} = \lim_{k \rightarrow \infty} \lambda_k^{(\alpha)} < \infty$ .

Moreover,  $(\lambda_\infty^{(\alpha)} - \lambda_k^{(\alpha)}) k^{\alpha-1}$  is bounded in  $k$ .

The preceding results treat the first term in (1.8). Now we pass to the second. By [30, Lemma 4.4], the functional  $\gamma \mapsto X_\gamma$  is a continuous mapping  $H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \rightarrow L^\infty(W_0)$  satisfying  $\|X_\gamma\|_{L^\infty(W_0)} \leq 3 \|\dot{\gamma}\|_{L^2}^2$ . Furthermore, if  $\gamma \in H_{\text{ia}}^2$ , there is some  $\beta_\gamma > 0$  continuously depending on  $\gamma$  such that  $0 \leq wX_\gamma(t, w) \leq 1 - \beta_\gamma$ . We conclude

**Corollary 1.5** The functional  $\gamma \mapsto \Omega_\gamma^{(\alpha)}$  is a continuous mapping

$$H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \rightarrow L^\infty(W_0).$$

Now we collect all the facts proved in this section.

**Theorem 1.6** ( $E^{(\alpha)} \in C^0(H_{\text{ir}}^2)$ ) The functional  $E^{(\alpha)}$  is continuous on  $H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .

*Proof.* The continuity on  $H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  follows by Propositions 1.2, 1.3, and Corollary 1.5. Note that  $\int_{-\pi}^\pi |w|^{2-\alpha} dw < \infty$ . Let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence of  $H_{\text{ir}}^2$  functions converging to some  $\gamma_0 \in H_{\text{ir}}^2$  with respect to the  $H^2$  norm. Using the continuity of the rescaling and reparametrizing operator  $\sim$  from Theorem A.1 and recalling  $\tilde{E}^{(\alpha)} \circ \sim = \tilde{E}^{(\alpha)}$ , we arrive at  $\tilde{E}^{(\alpha)}(\gamma_n) = \tilde{E}^{(\alpha)}(\tilde{\gamma}_n) \xrightarrow{n \rightarrow \infty} \tilde{E}^{(\alpha)}(\tilde{\gamma}_0) = \tilde{E}^{(\alpha)}(\gamma_0)$ . By the continuity of the length functional and the fact that  $(\mathcal{L}(\gamma_n))_{n \in \mathbb{N} \cup \{0\}}$  is bounded below by some positive constant, we conclude

$$E^{(\alpha)}(\gamma_n) = \mathcal{L}(\gamma_n)^{2-\alpha} \tilde{E}^{(\alpha)}(\gamma_n) \xrightarrow{n \rightarrow \infty} \mathcal{L}(\gamma_0)^{2-\alpha} \tilde{E}^{(\alpha)}(\gamma_0) = E^{(\alpha)}(\gamma_0).$$

□

## 1.2 First variation of $E^{(\alpha)}$

In order to shorten notation, we will use the notation  $\Delta \bullet := \bullet(t+w) - \bullet(t)$  from now on throughout this text.

**Lemma 1.7** ( $H_{\text{ir}}^2$  is open in  $H^2$ ) For any  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  there is some  $\delta_\gamma > 0$  such that  $B_{\delta_\gamma}(\gamma) \subset H_{\text{ir}}^2$ . More precisely, there is some  $\delta'_\gamma > 0$ , such that  $\gamma + h \in H_{\text{ir}}^2$  for all  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  satisfying  $\|\dot{h}\|_{L^\infty} \leq \delta'_\gamma$ .

*Proof.* The regularity of  $\gamma$  yields  $c := \min_{\mathbb{R}/2\pi\mathbb{Z}} |\dot{\gamma}| > 0$ . Moreover,  $\|\dot{\gamma}\|_{L^2} > 0$ , for  $\gamma$  is closed. Let  $\varepsilon := \left(\frac{1}{4}c\|\dot{\gamma}\|_{L^2}^{-1}\right)^2 > 0$ ,  $\varrho := \min_{(\bar{t}, \bar{w}) \in W_\varepsilon} |\gamma(\bar{t} + \bar{w}) - \gamma(\bar{t})| > 0$  by injectivity of  $\gamma$  (or  $\varrho := 1$  if  $\varepsilon > \pi$ ), and  $\delta'_\gamma := \frac{1}{2} \min(c, \frac{\varrho}{\pi}) > 0$ . Let  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  satisfy  $\|\dot{h}\|_{L^\infty} \leq \delta'_\gamma$ . If  $|w| \leq \varepsilon$ , we obtain  $|\Delta(\gamma + h)| \geq (c - \sqrt{\varepsilon}\|\dot{\gamma}\|_{L^2} - \delta'_\gamma)|w| \geq \frac{c}{4}|w| > 0$  if  $w \neq 0$ . On the other hand, from  $|w| \geq \varepsilon$  we deduce  $|\Delta(\gamma + h)| \geq \varrho - \pi \|\dot{h}\|_{L^\infty} \geq \frac{1}{2}\varrho > 0$ . This proves injectivity. Regularity is due to  $|(\gamma + h)'| \geq c - \|\dot{h}\|_{L^\infty} \geq \frac{1}{2}c > 0$ . By embedding inequalities we obtain  $\delta_\gamma$  from  $\delta'_\gamma$ . □

Using the notation from the preceding proof, we define the variable

$$\delta_0 = \delta_0\left(\gamma, \|\dot{h}\|_{L^\infty}\right) := \frac{\delta'_\gamma}{\|\dot{h}\|_{L^\infty}} = \frac{\min(c, \frac{\varrho}{\pi})}{2\|\dot{h}\|_{L^\infty}} > 0 \text{ if } h \neq \text{const} \quad \text{or} \quad \delta_0 := 1 \text{ otherwise,}$$

which will be used throughout this paragraph.



**Corollary 1.8** For any  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $\tau \in [-\delta_0, \delta_0]$  the curve  $\gamma_\tau := \gamma + \tau h$  still belongs to  $H_{\text{ir}}^2$ .

**Remark 1.9 (Injectivity and isotopy)** For any  $\gamma \in C_{\text{ir}}^1$ , there is an open  $C^1$ -neighbourhood which entirely consists of curves being ambient isotopic to  $\gamma$  and hence injective (cf. e. g. [39], where an explicit isotopy is constructed).

**Lemma 1.10** For any  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $\varepsilon \in (0, \pi)$ , and  $(t, w) \in W_\varepsilon$ , the mapping  $\tau \mapsto f^{(\alpha)}(\gamma_\tau; t, w)$  belongs to  $C^{0,1}([-\delta_0, \delta_0])$ . Therefore, it is a. e. differentiable, and its derivative amounts to  $\frac{d}{d\tau} f^{(\alpha)}(\gamma_\tau; t, w) = g^{(\alpha)}(\gamma_\tau, h; t, w)$ , where

$$\begin{aligned} g^{(\alpha)}(\gamma, h; t, w) &:= \alpha g_1^{(\alpha)} + g_2^{(\alpha)} + g_{2^\bullet}^{(\alpha)}, \\ g_1^{(\alpha)}(\gamma, h; t, w) &:= \left( \frac{d}{d\lambda} \Big|_{\lambda=0} \frac{D_{\gamma_\lambda}(t, t+w)}{D_\gamma(t, t+w)^{\alpha+1}} - \frac{\langle \Delta\gamma, \Delta h \rangle_{\mathbb{R}^d}}{|\Delta\gamma|^{\alpha+2}} \right) |\dot{\gamma}(t+w)| |\dot{\gamma}(t)|, \\ g_2^{(\alpha)}(\gamma, h; t, w) &:= \left( \frac{1}{|\Delta\gamma|^\alpha} - \frac{1}{D_\gamma(t, t+w)^\alpha} \right) \left\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \right\rangle_{\mathbb{R}^d} |\dot{\gamma}(t+w)| |\dot{\gamma}(t)|, \\ g_{2^\bullet}^{(\alpha)}(\gamma, h; t, w) &:= \left( \frac{1}{|\Delta\gamma|^\alpha} - \frac{1}{D_\gamma(t, t+w)^\alpha} \right) \left\langle \frac{\dot{\gamma}(t+w)}{|\dot{\gamma}(t+w)|^2}, \dot{h}(t+w) \right\rangle_{\mathbb{R}^d} |\dot{\gamma}(t+w)| |\dot{\gamma}(t)|. \end{aligned}$$

**Proof.** By the proof of Lemma 1.7, the mapping  $\tau \mapsto |\Delta\gamma_\tau|$  is uniformly bounded below on  $[-\delta_0, \delta_0] \times W_\varepsilon$ , so  $\tau \mapsto |\Delta\gamma_\tau|^{-\alpha}$  belongs to  $C^1([-\delta_0, \delta_0])$ . By

$$|\dot{\gamma}_\tau| \geq c - \delta_0 \left\| \dot{h} \right\|_{L^\infty} \geq \frac{1}{2}c, \quad \text{for } \delta_0 \leq \frac{c}{2 \left\| \dot{h} \right\|_{L^\infty}}, \quad (1.14)$$

also the mappings  $\tau \mapsto |\dot{\gamma}_\tau(t)|$  and  $\tau \mapsto |\dot{\gamma}_\tau(t+w)|$  are  $C^1([-\delta_0, \delta_0])$ . Since  $\tau \mapsto \mathcal{L}(\gamma_\tau|_{[t, t+w]}) = |w| \int_0^1 |\dot{\gamma}_\tau(t + \vartheta w)| d\vartheta$  is  $C^1([-\delta_0, \delta_0])$  with derivative  $\tau \mapsto \int_0^1 \left\langle \frac{\dot{\gamma}(t+\vartheta w)}{|\dot{\gamma}(t+\vartheta w)|}, \dot{h}(t + \vartheta w) \right\rangle d\vartheta$ , we deduce that  $\tau \mapsto D_{\gamma_\tau}(t, t+w)$  is  $C^{0,1}([-\delta_0, \delta_0])$  and bounded below on  $W_\varepsilon$  (since  $D_\gamma(t, t+w) \geq |\Delta\gamma|$ ). So, its  $(-\alpha)$ -th power is also Lipschitz (since  $|w| \geq \varepsilon > 0$ ).

Now the claim follows by the product rule for absolutely continuous functions.  $\square$

The preceding proof also shows  $g^{(\alpha)}(\gamma_\bullet, h; \cdot, \cdot) \in L^\infty([-\delta_0, \delta_0] \times W_\varepsilon)$  for  $\varepsilon \in (0, \pi)$ , which implies by LEBESGUE's theorem

**Corollary 1.11** For any  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $\varepsilon \in (0, \pi)$ , and  $(t, w) \in W_\varepsilon$ , both the mappings

$$\tau \mapsto I_\varepsilon^{(\alpha)}(\gamma_\tau, h) := \iint_{W_\varepsilon} g^{(\alpha)}(\gamma_\tau, h; t, w) dw dt \quad \text{and} \quad \tau \mapsto \iint_{W_\varepsilon} \left| g^{(\alpha)}(\gamma_\tau, h; t, w) \right| dw dt$$

belong to  $C^0([-\delta_0, \delta_0])$ .

To prepare the computation of the derivative of  $f^{(\alpha)}$  on  $W_\varepsilon$ , we first study  $D_{\gamma_\tau}$  near  $w = 0$ .

**Remark 1.12 (Differentiability of  $D_{\gamma_\tau}$ )** For each  $\gamma, h \in C^1$ ,  $|\dot{\gamma}| > 0$ , there is some  $\bar{\delta} > 0$ , such that, for any  $(\bar{\tau}; \bar{t}, \bar{w}) \in [-\bar{\delta}, \bar{\delta}] \times W_0$ , the mapping  $\tau \mapsto D_{\gamma_\tau}(\bar{t}, \bar{t} + w)$  is differentiable at  $\tau = \bar{\tau}$  up to at most one point  $w = \bar{w}(\bar{\tau}; \bar{t}) \in (-\pi, \pi)$ . (Note that  $\tau \mapsto D_{\gamma_\tau}(\bar{t}, \bar{t} + w)|_{w=0} \equiv 0$  is differentiable.) This relies on the fact that  $D_\gamma$  is defined as the minimum of two  $C^1([-\bar{\delta}, \bar{\delta}] \times W_0)$  functions, one of them strictly monotonically increasing in  $w$  and the other one monotonically decreasing in  $w$ .

In the subsequent Lemma we prove that  $\bar{w}(\bar{\tau}; \bar{t})$  is uniformly bounded away from  $w = 0$  such that  $(\tau \mapsto D_{\gamma_\tau}) \in C^1$  in some neighbourhood of  $(\tau; t, w) \in \{0\} \times \mathbb{R}/2\pi\mathbb{Z} \times \{0\}$ .

**Lemma 1.13** For any  $\gamma, h \in C^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $|\dot{\gamma}| \geq c > 0$  we define  $\bar{\delta} = \bar{\delta}(\gamma, \|\dot{h}\|_{L^\infty}) := \frac{1}{2}c \|\dot{h}\|_{L^\infty}^{-1} > 0$  if  $h \not\equiv \text{const}$  or  $\bar{\delta} := 1$  otherwise, and  $\bar{\varepsilon} = \bar{\varepsilon}(\gamma) := \pi \left(\frac{2}{c} \|\dot{\gamma}\|_{L^\infty} + 1\right)^{-1}$ . Then, for all  $(\tau; t, w) \in [-\bar{\delta}, \bar{\delta}] \times \mathbb{R}/2\pi\mathbb{Z} \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ ,

$$D_{\gamma_\tau}(t, t+w) \stackrel{!}{=} \mathcal{L}(\gamma_\tau|_{[t, t+w]}) = \left| \int_t^{t+w} |\dot{\gamma}_\tau(\theta)| d\theta \right| = |w| \int_0^1 |\dot{\gamma}_\tau(t + \vartheta w)| d\vartheta. \quad (1.15)$$

Clearly, for each pair  $(t, w) \in \mathbb{R}/2\pi\mathbb{Z} \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ , the mapping  $\tau \mapsto D_{\gamma_\tau}(t, t+w)$  is  $C^1([-\bar{\delta}, \bar{\delta}])$ , and its derivative amounts to

$$\frac{d}{d\tau} D_{\gamma_\tau}(t, t+w) = |w| \int_0^1 \left\langle \frac{\dot{\gamma}_\tau(t + \vartheta w)}{|\dot{\gamma}_\tau(t + \vartheta w)|}, \dot{h}(t + \vartheta w) \right\rangle_{\mathbb{R}^d} d\vartheta. \quad (1.16)$$

**Proof.** Equation (1.15) can be directly derived by verifying  $\mathcal{L}(\gamma_\tau|_{[t, t+w]}) \leq \mathcal{L}(\gamma_\tau) - \mathcal{L}(\gamma_\tau|_{[t, t+w]})$  for  $(\tau; t, w) \in [-\bar{\delta}, \bar{\delta}] \times \mathbb{R}/2\pi\mathbb{Z} \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ . Since  $\tau \mapsto |\dot{\gamma}_\tau(t + \vartheta w)|$  is  $C^1([-\bar{\delta}, \bar{\delta}])$  by (1.14) with derivative  $\tau \mapsto \left\langle \frac{\dot{\gamma}(t + \vartheta w)}{|\dot{\gamma}(t + \vartheta w)|}, \dot{h}(t + \vartheta w) \right\rangle$ , the difference quotient  $\frac{1}{\lambda} (|\dot{\gamma}_{\tau+\lambda}(t + \vartheta w)| - |\dot{\gamma}_\tau(t + \vartheta w)|)$  is bounded by  $\sup_{\tau \in [-\bar{\delta}, \bar{\delta}]} \frac{d}{d\tau} |\dot{\gamma}_\tau(t + \vartheta w)| \leq \|\dot{h}\|_{L^\infty}$  due to the mean value theorem. Using LEBESGUE's theorem, we may interchange differentiation and integration. Since the right-hand side of (1.16) continuously depends on  $\tau$  (again by LEBESGUE's theorem), the second claim follows.  $\square$

The proof of the following lemma is straightforward but quite long, so it will be omitted. The essential technique is applying suitable Taylor expansions of the quantities appearing in the integrand of  $I^{(\alpha)}$ . The details can be found in [40, pp. 42 – 48].

**Lemma 1.14** Let  $\gamma \in C_{\text{ir}}^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in C^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . Then the mapping

$$\tau \mapsto I^{(\alpha)}(\gamma_\tau, h) := \lim_{\varepsilon \searrow 0} I_\varepsilon^{(\alpha)}(\gamma_\tau, h)$$

is well-defined and continuous on  $[-\delta_0, \delta_0]$ . Moreover, there is some  $\varepsilon_1 = \varepsilon_1(\gamma, h) > 0$  such that

$$\sup \left\{ \left| I_\varepsilon^{(\alpha)}(\gamma_\tau, h) \right| \mid \varepsilon \in (0, \varepsilon_1], \tau \in [-\delta_0, \delta_0] \right\} < \infty.$$

**Lemma 1.15** Let  $\gamma \in C_{\text{ir}}^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in C^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . Then  $E^{(\alpha)}$  is differentiable at  $\gamma$  in direction  $h$ . Its first variation amounts to  $\delta E^{(\alpha)}(\gamma, h) = I^{(\alpha)}(\gamma, h)$ .

**Proof.** For  $\varepsilon \in (0, \pi)$ , we deduce  $\frac{f^{(\alpha)}(\gamma_\tau; t, w) - f^{(\alpha)}(\gamma; t, w)}{\tau} = \int_0^1 g^{(\alpha)}(\gamma + \vartheta\tau h, h; t, w) d\vartheta$  on  $(\tau; t, w) \in [-\delta_0, \delta_0] \times W_\varepsilon$  from Lemma 1.10, where we also used the linearity of  $g^{(\alpha)}(\gamma_\tau, \cdot; t, w)$ . Moreover, Corollary 1.11 permits to use FUBINI's theorem, so

$$\begin{aligned} & \frac{\iint_{W_\varepsilon} f^{(\alpha)}(\gamma_\tau; t, w) dw dt - \iint_{W_\varepsilon} f^{(\alpha)}(\gamma; t, w) dw dt}{\tau} \\ &= \iint_{W_\varepsilon} \frac{f^{(\alpha)}(\gamma_\tau; t, w) - f^{(\alpha)}(\gamma; t, w)}{\tau} dw dt = \iint_{W_\varepsilon} \int_0^1 g^{(\alpha)}(\gamma + \vartheta\tau h, h; t, w) d\vartheta dw dt \\ &= \int_0^1 \iint_{W_\varepsilon} g^{(\alpha)}(\gamma + \vartheta\tau h, h; t, w) dw dt d\vartheta = \int_0^1 I_\varepsilon^{(\alpha)}(\gamma + \vartheta\tau h, h) d\vartheta. \end{aligned}$$

Applying LEBESGUE's theorem on monotone convergence (recall  $f^{(\alpha)} \geq 0$ ) to the left-hand side and LEBESGUE's theorem on dominated convergence to the right-hand side, we may pass to the limit  $\varepsilon \searrow 0$  arriving at

$$\frac{E^{(\alpha)}(\gamma_\tau) - E^{(\alpha)}(\gamma)}{\tau} = \int_0^1 I^{(\alpha)}(\gamma + \vartheta\tau h, h) d\vartheta.$$

By Lemma 1.14, the right-hand side converges as  $\tau \rightarrow 0$ .  $\square$

The identity  $g^{(\alpha)}(\varrho\gamma, \varrho h; t, w) = \varrho^{2-\alpha} g^{(\alpha)}(\gamma, h; t, w)$  for  $\varrho > 0$ , a. e.  $(t, w) \in W_\varepsilon$  gives positive homogeneity of  $I_\varepsilon^{(\alpha)}$  which transfers this property to  $I^{(\alpha)}$ . By the same argument one obtains invariance under orthogonal transformations and dilatations. On the other hand, applying the transformation rule (A.1) changes the integration domain  $W_\varepsilon$  which leads to some difficulties. Using the preceding lemma and recalling the invariance under parametrization of  $E^{(\alpha)}$ , we just obtain an easy proof for invariance under parametrization.

**Corollary 1.16** *The functional  $I^{(\alpha)}$  is parametrization (and orientation) invariant on  $C_{\text{ir}}^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \times C^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ , so it is completely determined by its values on  $C_{\text{ia}}^3 \times C^3$ . More precisely, let  $(\gamma, h) \in C_{\text{ir}}^3 \times C^3$  and  $\Psi\gamma$  as in Proposition A.6. Then*

$$(\tilde{\gamma}, \tilde{h}) := \frac{2\pi}{\mathcal{L}(\gamma)}(\gamma, h) \circ (\Psi\gamma)^{-1} \in C_{\text{ia}}^3 \times C^3 \quad (1.17)$$

and

$$I^{(\alpha)}(\gamma, h) = \left( \frac{2\pi}{\mathcal{L}(\gamma)} \right)^{\alpha-2} I^{(\alpha)}(\tilde{\gamma}, \tilde{h}). \quad (1.18)$$

### 1.3 Continuity of $\delta E^{(\alpha)}$ on $H_{\text{ir}}^2 \times H^2$

For this section we partially use arguments from [30, Lemma 4.1 – 4.3]. We start by deriving a slightly simpler formula for  $I_\varepsilon^{(\alpha)}$  on  $H_{\text{ir}}^2 \times H^2$ . Since  $\iint_{W_\varepsilon} g_{2^\bullet}^{(\alpha)}(\gamma, h; \cdot, \cdot) = \iint_{W_\varepsilon} g_2^{(\alpha)}(\gamma, h; \cdot, \cdot)$ , we arrive at

$$\begin{aligned} I_\varepsilon^{(\alpha)}(\gamma, h) &= \alpha \iint_{W_\varepsilon} \left( \frac{\frac{d}{d\lambda} \Big|_{\lambda=0} D_{\gamma\lambda}(t, t+w)}{D_\gamma(t, t+w)^{\alpha+1}} - \frac{\langle \Delta\gamma, \Delta h \rangle_{\mathbb{R}^d}}{|\Delta\gamma|^{\alpha+2}} \right) |\dot{\gamma}(t+w)| |\dot{\gamma}(t)| dw dt \\ &\quad + 2 \iint_{W_\varepsilon} \left( \frac{1}{|\Delta\gamma|^\alpha} - \frac{1}{D_\gamma(t, t+w)^\alpha} \right) \left\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \right\rangle_{\mathbb{R}^d} |\dot{\gamma}(t+w)| |\dot{\gamma}(t)| dw dt. \end{aligned} \quad (1.19)$$

Restricting to curves parametrized by arc-length, we even obtain a shorter formula which no longer involves the functional  $D_\gamma$  or its derivative.

**Lemma 1.17** *The functional  $I^{(\alpha)}$ , which was defined as  $\lim_{\varepsilon \searrow 0} I_\varepsilon^{(\alpha)}$  on  $C_{\text{ir}}^3 \times C^3$  in Lemma 1.14, is continuous on  $(H_{\text{ia}}^2 \times H^2)(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ , where*

$$I^{(\alpha)}(\gamma, h) = \lim_{\varepsilon \searrow 0} \iint_{W_\varepsilon} \left\{ (\alpha-2) \frac{\langle \dot{\gamma}(t), \dot{h}(t) \rangle}{|w|^\alpha} + 2 \frac{\langle \dot{\gamma}(t), \dot{h}(t) \rangle}{|\Delta\gamma|^\alpha} - \alpha \frac{\langle \Delta\gamma, \Delta h \rangle}{|\Delta\gamma|^{\alpha+2}} \right\} dw dt. \quad (1.20)$$

*Proof.* Let  $\gamma \in H_{\text{ia}}^2$ ,  $h \in H^2$ , and  $(t, w) \in W_0$ . We start with the computation of  $D_\gamma$  and  $\frac{d}{d\tau} \Big|_{\tau=0} D_{\gamma_\tau}$ . Of course,  $D_\gamma(t, t+w) = |w|$ . Let  $\tilde{\delta}_h(w) := \frac{1}{2\pi}(\pi - |w|) \left\| \dot{h} \right\|_{L^\infty}^{-1}$  if  $h \neq \text{const}$  and  $\tilde{\delta}_h(w) := 1$  otherwise. Let  $w \in (-\pi, \pi)$  which gives  $\tilde{\delta}_h(w) > 0$ . If now  $|\tau| \leq \tilde{\delta}_h(w)$ , we obtain  $\mathcal{L}(\gamma_\tau|_{[t, t+w]}) \leq \frac{1}{2} \mathcal{L}(\gamma_\tau)$  as in the proof of Lemma 1.13. This implies (1.15) and (1.16); note that  $C^3$ -regularity is not necessary here. We conclude, for any pair  $(t, w) \in \mathbb{R}/2\pi\mathbb{Z} \times (-\pi, \pi)$ ,

$$\frac{d}{d\tau} \Big|_{\tau=0} D_{\gamma_\tau}(t, t+w) = |w| \int_0^1 \left\langle \dot{\gamma}(t+\vartheta w), \dot{h}(t+\vartheta w) \right\rangle d\vartheta.$$

Applying FUBINI's Theorem twice, we arrive at

$$\iint_{W_\varepsilon} \frac{\frac{d}{d\tau} \Big|_{\tau=0} D_{\gamma_\tau}(t, t+w)}{D_\gamma(t, t+w)^{\alpha+1}} dw dt = \iint_{W_\varepsilon} \frac{\langle \dot{\gamma}(\bar{t}), \dot{h}(\bar{t}) \rangle}{|w|^\alpha} dw d\bar{t}. \quad (1.21)$$

So we obtain (1.20) from (1.19). Now, using (1.6),

$$\begin{aligned} I_\varepsilon^{(\alpha)}(\gamma, h) - \alpha Q_\varepsilon^{(\alpha)}(\gamma, h) &= -\alpha \iint_{W_\varepsilon} \langle \Delta\gamma, \Delta h \rangle \left( \frac{1}{|\Delta\gamma|^{\alpha+2}} - \frac{1}{|w|^{\alpha+2}} \right) dw dt + \\ &\quad + 2 \iint_{W_\varepsilon} \langle \dot{\gamma}(t), \dot{h}(t) \rangle \left( \frac{1}{|\Delta\gamma|^\alpha} - \frac{1}{|w|^\alpha} \right) dw dt. \end{aligned}$$

The identities  $\frac{1}{|\Delta\gamma|^\alpha} - \frac{1}{|w|^\alpha} = \frac{\alpha}{2} \frac{w}{|w|^\alpha} X_\gamma + \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \Omega_\gamma^{(\alpha)}$  and  $\frac{1}{|\Delta\gamma|^2} - \frac{1}{w^2} = \frac{X_\gamma}{w} + \frac{X_\gamma^2}{1-wX_\gamma}$  yield

$$\begin{aligned} &\langle \Delta\gamma, \Delta h \rangle \left( \frac{1}{|\Delta\gamma|^{\alpha+2}} - \frac{1}{|w|^{\alpha+2}} \right) \\ &= \frac{\langle \Delta\gamma, \Delta h \rangle}{w^2} \left[ \left( \frac{\alpha}{2} + 1 \right) \frac{w}{|w|^\alpha} X_\gamma + \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} \frac{1}{|w|^{\alpha-2}} X_\gamma^2 + \frac{w}{|w|^{\alpha-2}} X_\gamma^3 \Omega_\gamma^{(\alpha)} \right. \\ &\quad \left. + \frac{X_\gamma^2}{1-wX_\gamma} \left( \frac{\alpha}{2} \frac{w}{|w|^{\alpha-2}} X_\gamma + |w|^{4-\alpha} X_\gamma^2 \Omega_\gamma^{(\alpha)} + \frac{1}{|w|^{\alpha-2}} \right) \right]. \end{aligned}$$

So,

$$\begin{aligned} &I_\varepsilon^{(\alpha)}(\gamma, h) - \alpha Q_\varepsilon^{(\alpha)}(\gamma, h) \\ &= \alpha \iint_{W_\varepsilon} \frac{1}{|w|^{\alpha-2}} \cdot \frac{X_\gamma}{w} \left[ \langle \dot{\gamma}(t), \dot{h}(t) \rangle - \left( \frac{\alpha}{2} + 1 \right) \frac{\langle \Delta\gamma, \Delta h \rangle}{w^2} \right] dw dt \\ &\quad + 2 \iint_{W_\varepsilon} \langle \dot{\gamma}(t), \dot{h}(t) \rangle \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \Omega_\gamma^{(\alpha)} dw dt \tag{1.22} \\ &\quad - \alpha \iint_{W_\varepsilon} \frac{\langle \Delta\gamma, \Delta h \rangle}{w^2} \left[ \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} + \frac{1}{1-wX_\gamma} \right) \right. \\ &\quad \left. + \frac{w}{|w|^{\alpha-2}} X_\gamma^3 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} \frac{1}{1-wX_\gamma} \right) + |w|^{4-\alpha} \frac{X_\gamma^4}{1-wX_\gamma} \Omega_\gamma^{(\alpha)} \right] dw dt. \end{aligned}$$

By Proposition 1.3, the functional  $Q^{(\alpha)} = \lim_{\varepsilon \searrow 0} Q_\varepsilon^{(\alpha)}$  is continuous on  $H^2 \times H^2$ . We now have to treat the remaining three terms on the right-hand side of (1.22).

By Corollary 1.5, the operators  $X$ ,  $\Omega^{(\alpha)}$ , and  $\frac{1}{1-wX}$  are continuous on  $H_{\text{ia}}^2$ -sequences with respect to the  $L^\infty$  norm. Moreover we deduce from (1.4)

$$\begin{aligned} \frac{X_\gamma(t, w)}{w} &= 2 \int_0^1 u(1-u) \left\langle \int_0^1 \ddot{\gamma}(t+uvw) dv, \ddot{\gamma}(t+uw) \right\rangle_{\mathbb{R}^d} du \\ &\quad - \int_0^1 \int_0^1 (1-u)(1-v) \langle \ddot{\gamma}(t+uw), \ddot{\gamma}(t+vw) \rangle_{\mathbb{R}^d} du dv, \end{aligned} \tag{1.23}$$

which is obviously (bounded and) continuous on  $H^2$ , again with respect to the  $L^\infty$  norm. The same holds true for

$$\frac{\langle \Delta\gamma, \Delta h \rangle}{w^2} = \left\langle \int_0^1 \dot{\gamma}(t+\sigma w) d\sigma, \int_0^1 \dot{h}(t+\tau w) d\tau \right\rangle_{\mathbb{R}^d}, \tag{1.24}$$

and  $\langle \dot{\gamma}(t+\cdot), \dot{h}(t+\cdot) \rangle$  on  $(\gamma, h) \in H^2 \times H^2$ . Now  $\int_{-\pi}^\pi \frac{dw}{|w|^{\alpha-2}} < \infty$  permits to pass to the limit  $\varepsilon \searrow 0$  which concludes the proof.  $\square$

**Lemma 1.18** *The mapping*

$$(\gamma, h) \mapsto \left( \frac{2\pi}{\mathcal{L}(\gamma)} \right)^{\alpha-2} I^{(\alpha)} \left( \frac{2\pi}{\mathcal{L}(\gamma)} (\gamma, h) \circ (\Psi\gamma)^{-1} \right), \quad (1.25)$$

where  $\Psi\gamma$  is taken from Proposition A.6, defines a continuous extension of  $I^{(\alpha)}$  to  $(H_{\text{ir}}^2 \times H^2)(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .

**Remark 1.19** Note that, while  $I^{(\alpha)}$  can directly be computed by (1.20) on  $H_{\text{ia}}^2 \times H^2$ , we cannot use this formula for  $H_{\text{ir}}^2 \times H^2$ , but have to pass to (1.25) instead.

**Proof.** If  $(\gamma, h) \in C_{\text{ir}}^3 \times C^3$ , the statement follows from Corollary 1.16 (invariance under reparametrization). Now we consider  $(\gamma, h) \in H_{\text{ir}}^2 \times H^2$ . By Proposition A.6 and Corollary A.4, the mapping  $\gamma \mapsto (\Psi\gamma)^{-1} \in H_{\text{p}}^2(0, \ell)$ , see (A.3), is continuous with respect to the  $H^2$  topology as well as  $\gamma \mapsto \mathcal{L}(\gamma)$ . So  $(\gamma, h) \mapsto \left( \frac{2\pi}{\mathcal{L}(\gamma)} (\gamma, h) \circ (\Psi\gamma)^{-1} \right) \in H_{\text{ia}}^2 \times H^2$  is continuous with respect to the  $H^2 \times H^2$  topology. The claim follows by Lemma 1.17.  $\square$

**Lemma 1.20 (Approximation in  $H_{\text{ir}}^2 \times H^2$ )** For any  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  there are sequences  $(\gamma_k)_{k \in \mathbb{N}} \subset C_{\text{ir}}^\infty$ ,  $(h_k)_{k \in \mathbb{N}} \subset C^\infty$  converging to  $\gamma$  and  $h$  in  $H^2$  and some  $\delta_1 = \delta_1(\gamma, \|h\|_{H^2}) > 0$  such that  $\gamma_k + \tau h_k \in C_{\text{ir}}^\infty$  for all  $\tau \in [-\delta_1, \delta_1]$ .

**Proof.** Take  $\delta_\gamma = \delta_\gamma(\gamma) > 0$  from Lemma 1.7. Let  $(\gamma_k)_{k \in \mathbb{N}}, (h_k)_{k \in \mathbb{N}}$  be sequences approximating  $\gamma, h$  in  $H^2$ . By dropping some elements (if necessary) we obtain  $\|\gamma_k - \gamma\|_{H^2} \leq \frac{1}{3}\delta_\gamma$  (which implies  $(\gamma_k)_{k \in \mathbb{N}} \subset C_{\text{ir}}^\infty$ ) and  $\|h_k - h\|_{H^2} \leq \frac{1}{3}\delta_\gamma$  for all  $k \in \mathbb{N}$ . Assuming  $\tau \in [-\delta_1, \delta_1]$ , where  $\delta_1 = \delta_1(\gamma, \|h\|_{H^2}) := \min\left(1, \frac{1}{3}\delta_\gamma \|h\|_{H^2}^{-1}\right)$  if  $h \neq 0$  or  $\delta_1 := 1$  otherwise, we arrive at  $\|(\gamma_k + \tau h_k) - \gamma\|_{H^2} \leq \|\gamma_k - \gamma\|_{H^2} + \delta_1 \|h\|_{H^2} + \delta_1 \|h_k - h\|_{H^2} \leq \delta_\gamma$ . So  $\gamma_k + \tau h_k$  belongs to  $C_{\text{ir}}^\infty$  for all  $k \in \mathbb{N}$ .  $\square$

**Theorem 1.21 ( $\delta E^{(\alpha)} \in C^0(H_{\text{ir}}^2 \times H^2)$ )** Let  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . Then  $E^{(\alpha)}$  is Fréchet differentiable at  $\gamma$  in direction of  $h$ , and its differential (first variation) amounts to  $\delta E^{(\alpha)}(\gamma, h) = I^{(\alpha)}(\gamma, h)$ . More precisely, the derivative  $h \mapsto \delta E^{(\alpha)}(\gamma, h)$  is linear and continuous on  $H^2$  and satisfies

$$\frac{E^{(\alpha)}(\gamma + h) - E^{(\alpha)}(\gamma) - \delta E^{(\alpha)}(\gamma, h)}{\|h\|_{H^2}} \xrightarrow{h \rightarrow 0} 0.$$

**Proof.** Let  $(\gamma_k)_{k \in \mathbb{N}}, (h_k)_{k \in \mathbb{N}} \subset C^\infty$  be sequences approximating  $\gamma \in H_{\text{ir}}^2$  and  $h \in H^2$  as in Lemma 1.20. This admits to write using Lemma 1.15

$$\frac{E^{(\alpha)}(\gamma_k + \tau h_k) - E^{(\alpha)}(\gamma_k)}{\tau} = \frac{1}{\tau} \int_0^1 \frac{d}{d\vartheta} E^{(\alpha)}(\gamma_k + \vartheta \tau h_k) d\vartheta = \int_0^1 I^{(\alpha)}(\gamma_k + \vartheta \tau h_k, h_k) d\vartheta.$$

By Theorem 1.6 and Lemma 1.18 we may pass to the limit  $k \rightarrow \infty$  obtaining

$$\frac{E^{(\alpha)}(\gamma + \tau h) - E^{(\alpha)}(\gamma)}{\tau} = \int_0^1 I^{(\alpha)}(\gamma + \vartheta \tau h, h) d\vartheta,$$

which tends to  $\delta E^{(\alpha)}(\gamma, h) = I^{(\alpha)}(\gamma, h)$  as  $\tau \rightarrow 0$ . The integrand on the right-hand side is again majorized due to Lemma 1.18. The linearity of  $h \mapsto \delta E^{(\alpha)}(\gamma, h)$  is due to (1.20) and (1.25), its continuity follows from Lemma 1.18. Finally

$$\begin{aligned} & \left| E^{(\alpha)}(\gamma + h) - E^{(\alpha)}(\gamma) - \delta E^{(\alpha)}(\gamma, h) \right| \\ & \leq \|h\|_{H^2} \int_0^1 \left| I^{(\alpha)}\left(\gamma + \vartheta h, \frac{h}{\|h\|_{H^2}}\right) - I^{(\alpha)}\left(\gamma, \frac{h}{\|h\|_{H^2}}\right) \right| d\vartheta. \end{aligned}$$

The integral vanishes as  $h \rightarrow 0$  according to Lemma 1.18, which also yields an integrable majorant.  $\square$

### 1.4 Derivative formulae

In this section we state some formulae for the first variation  $\delta E^{(\alpha)}$  and the derivative  $E^{(\alpha)'} = H^{(\alpha)}$  satisfying  $\delta E^{(\alpha)}(\gamma, h) = \int_0^{2\pi} \langle (H^{(\alpha)}\gamma)(t), h(t) \rangle_{\mathbb{R}^d} |\dot{\gamma}(t)| dt$ . These results are not required for the next section, so we omit the proofs, which can be found in [40, § 1.6]. By  $C_{\text{ir}}^{m,\beta}$  we denote the subset of  $C_{\text{ir}}^m$  consisting of functions which have a Hölder continuous  $m$ -th derivative with exponent  $\beta \in (0, 1]$ , i. e.  $|f^{(m)}(t+w) - f^{(m)}(w)| \leq C_f |w|^\beta$ .

**Proposition 1.22** *For  $(\gamma, h) \in H_{\text{ir}}^2 \times H^2$ , the first variation  $\delta E^{(\alpha)}(\gamma, h)$  is the limit of*

$$\iint_{D_{\gamma}(t, t+w) \geq \frac{\mathcal{L}(\gamma)}{2\pi} \varepsilon} \left\{ (\alpha - 2) \frac{\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \rangle}{D_{\gamma}(t, t+w)^\alpha} + 2 \frac{\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \rangle}{|\Delta\gamma|^\alpha} - \alpha \frac{\langle \Delta\gamma, \Delta h \rangle}{|\Delta\gamma|^{\alpha+2}} \right\} |\dot{\gamma}(t+w)| |\dot{\gamma}(t)| dw dt \quad (1.26)$$

as  $\varepsilon \searrow 0$ . If moreover  $(\gamma, h) \in C_{\text{ir}}^{2,1} \times C^{2,1}$ , then  $\delta E^{(\alpha)}(\gamma, h)$  equals

$$\lim_{\varepsilon \searrow 0} \iint_{W_\varepsilon} \left\{ (\alpha - 2) \frac{\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \rangle}{D_{\gamma}(t, t+w)^\alpha} + 2 \frac{\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \rangle}{|\Delta\gamma|^\alpha} - \alpha \frac{\langle \Delta\gamma, \Delta h \rangle}{|\Delta\gamma|^{\alpha+2}} \right\} |\dot{\gamma}(t+w)| |\dot{\gamma}(t)| dw dt. \quad (1.27)$$

Let  $\gamma \in C_{\text{ir}}^{3,\beta}$ ,  $\beta \in (\alpha - 2, 1]$ ,  $\lambda \in H^2(\mathbb{R}/2\pi\mathbb{Z})$ . HE [30, Lemma 4.6] provides a nice geometric argument which also works for  $\alpha \in [2, 3]$ , showing  $\delta E^{(\alpha)}(\gamma, \lambda\dot{\gamma}) = 0$ .

Approximating we obtain from the continuous extension of  $I^{(\alpha)}$  derived in Lemma 1.18

**Corollary 1.23** *Let  $\gamma \in H_{\text{ir}}^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $\lambda \in H^2(\mathbb{R}/2\pi\mathbb{Z})$ . Then  $\delta E^{(\alpha)}(\gamma, \lambda\dot{\gamma}) = 0$ .*

For the next statement we need the projection operator onto  $\dot{\gamma}^\perp$  given by  $P_{\dot{\gamma}^\perp} g := g - \left\langle g, \frac{\dot{\gamma}}{|\dot{\gamma}|} \right\rangle_{\mathbb{R}^d} \frac{\dot{\gamma}}{|\dot{\gamma}|}$ . Note that, provided  $|\dot{\gamma}| \equiv 1$ , this definition coincides with the other one given below in (2.5).

**Theorem 1.24** *Let  $\gamma \in C_{\text{ir}}^{3,1}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $h \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . Then  $\delta E^{(\alpha)}(\gamma, h)$  may be written as*

$$\delta E^{(\alpha)}(\gamma, h) = \int_0^{2\pi} \langle (H^{(\alpha)}\gamma)(t), h(t) \rangle_{\mathbb{R}^d} |\dot{\gamma}(t)| dt, \quad (1.28)$$

where  $(H^{(\alpha)}\gamma)(t) := \lim_{\varepsilon \searrow 0} \int_{|w| \in [\varepsilon, \pi]} \left\{ -(\alpha - 2) \frac{\frac{1}{|\dot{\gamma}(t)|} \langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \rangle}{D_{\gamma}(t, t+w)^\alpha} - 2 \frac{\frac{1}{|\dot{\gamma}(t)|} \langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \rangle}{|\Delta\gamma|^\alpha} + 2\alpha \frac{P_{\dot{\gamma}^\perp} \Delta\gamma}{|\Delta\gamma|^{\alpha+2}} \right\} |\dot{\gamma}(t+w)| dw$ .

**Remark 1.25** Equation (1.28) is the starting point for the gradient flow of O'HARA's energies, i. e. the solution to the *evolution equation*  $\frac{d}{dt}\gamma = -H^{(\alpha)}\gamma$ . For  $\alpha = 2$ , a short-time existence result requiring smooth initial data was derived in [30, Thm. 2.1]. It was improved in [6] to long-time existence for  $C^{2,\alpha}$ -initial data being close to a local minimizer; after suitable reparametrization the gradient flow converges smoothly to a (possibly different) local minimizer. A generalization to  $\alpha \in (2, 3)$  will appear in [7].

## 2 Smoothness of critical points

We start this section defining the Fourier multiplier  $\mathcal{M}_\mu$  and stating some embeddings. Results of this kind are usually proven by interpolation methods, cf. [27, Sect. 1.3], [51, Sect. 4.2], [52, p. 24], which may even give stronger results. (E. g., the assertion stated in Lemma 2.6 below also holds true for  $p = \frac{2}{1-2s}$ .) However, proofs can be given by elementary means, see [40, § 2.1].

We introduce two double sequences,

$$\sigma_{m,n} := \frac{\sqrt{1+m^2} - \sqrt{1+n^2}}{\sqrt{1+(m-n)^2}} \in (-1, 1), \quad \tau_{m,n} := \frac{1+n^2}{(1+m^2)(1+(m-n)^2)} \in (0, 2],$$

where  $m, n \in \mathbb{Z}$ . For  $\beta \geq 1$ , the first one extends to  $\sigma_{m,n}^{(\beta)} := \frac{(1+m^2)^{\beta/2} - (1+n^2)^{\beta/2}}{(1+(m-n)^2)^{\beta/2} (1+n^2)^{(\beta-1)/2}}$ , which is uniformly bounded by  $2^\beta$  as follows. By  $(1+z)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} z^k$  for  $|z| \leq 1$  and  $\binom{\beta}{k} = \mathcal{O}(k^{-\beta-1})$  as  $k \rightarrow \infty$  we

derive  $0 \leq (x+y)^\beta - y^\beta \leq 2^\beta x \max(x,y)^{\beta-1}$  for  $x, y \geq 0$ . Defining  $x := |\sqrt{1+m^2} - \sqrt{1+n^2}| \geq 0$  and  $y := \sqrt{1+n^2} \geq 1$  we arrive at  $(1+m^2)^{\beta/2} - (1+n^2)^{\beta/2} \leq 2^\beta (1+(m-n)^2)^{\beta/2} (1+\min(m,n)^2)^{(\beta-1)/2}$ . For the lower bound, we interchange the variables  $m, n$ .

Of course,  $\sigma_{m,n}^{(1)} = \sigma_{m,n}$ .

**Theorem 2.1 (Multiplier)** *Let  $(\mu_{k,l})_{k,l \in \mathbb{Z}} \in L^\infty(\mathbb{Z}^2, \mathbb{C})$  be a (uniformly bounded) double sequence satisfying  $\overline{\mu_{k,l}} = \mu_{-k,-l}$  for all  $k, l \in \mathbb{Z}$  and  $f, g \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ . Then*

$$\mathcal{M}_\mu(f, g) := \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \mu_{k, m-k} \hat{f}_k \hat{g}_{m-k} \phi_m \quad (2.1)$$

defines a bilinear operator with the following properties.

- (i) *If  $s > \frac{1}{2}$  and  $f, g \in H^s(\mathbb{R}/2\pi\mathbb{Z})$ , we obtain  $\mathcal{M}_\mu(f, g) \in H^s(\mathbb{R}/2\pi\mathbb{Z})$  satisfying  $\|\mathcal{M}_\mu(f, g)\|_{H^s} \leq C_s \|\mu\|_{\ell^\infty} \|f\|_{H^s} \|g\|_{H^s}$ .*
- (ii) *For any  $s \geq 0, \varepsilon > 0$  and  $f \in H^s(\mathbb{R}/2\pi\mathbb{Z}), g \in H^{s+\frac{1}{2}+\varepsilon}(\mathbb{R}/2\pi\mathbb{Z})$  we arrive at  $\mathcal{M}_\mu(f, g) \in H^s(\mathbb{R}/2\pi\mathbb{Z})$  and  $\|\mathcal{M}_\mu(f, g)\|_{H^s} \leq C_{s,\varepsilon} \|\mu\|_{\ell^\infty} \|f\|_{H^s} \|g\|_{H^{s+\frac{1}{2}+\varepsilon}}$ .*
- (iii) *For any  $s \geq 0, \varepsilon > 0$  and  $f, g \in H^s(\mathbb{R}/2\pi\mathbb{Z})$  we deduce  $J^{-\frac{1}{2}-\varepsilon} \mathcal{M}_\mu(f, g) \in H^s(\mathbb{R}/2\pi\mathbb{Z})$  and  $\|J^{-\frac{1}{2}-\varepsilon} \mathcal{M}_\mu(f, g)\|_{H^s} \leq C_{s,\varepsilon} \|\mu\|_{\ell^\infty} \|f\|_{H^s} \|g\|_{H^s}$ .*

The statement extends to vector or matrix valued functions  $f, g$  as well.

**Remark 2.2** Since  $\|f\|_{H^s} = \|J^s f\|_{L^2}$  for  $s \geq 0$ , we may take this identity as definition for  $\|f\|_{H^s}$  in case  $s < 0$ , such that

$$\|\mathcal{M}_\mu(f, g)\|_{H^s} \leq \begin{cases} C_s \|\mu\|_{\ell^\infty} \|f\|_{H^s} \|g\|_{H^s} & \text{if } s > \frac{1}{2}, \\ C_{s,\varepsilon} \|\mu\|_{\ell^\infty} \|f\|_{H^s} \|g\|_{H^{s+\frac{1}{2}+\varepsilon}} & \text{if } s \geq 0, \\ C_{s,\varepsilon} \|\mu\|_{\ell^\infty} \|f\|_{H^{s+\frac{1}{2}+\varepsilon}} \|g\|_{H^{s+\frac{1}{2}+\varepsilon}} & \text{if } s \geq -\frac{1}{2} - \varepsilon, \quad \varepsilon > 0. \end{cases}$$

Applying Theorem 2.1 to  $\mu_{k,\ell} \equiv 1$  leads to  $\mathcal{M}_1(f, g) = fg$ , which gives

**Corollary 2.3** ( $H^s \cdot H^s \hookrightarrow H^s$ )  $\begin{cases} s > \frac{1}{2} \rightsquigarrow \|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s}, \\ s \geq 0 \rightsquigarrow \|fg\|_{H^s} \leq C_{s,\varepsilon} \|f\|_{H^s} \|g\|_{H^{s+\frac{1}{2}+\varepsilon}}. \end{cases}$

**Lemma 2.4** ( $L^1 \hookrightarrow H^{-1/2-\varepsilon}$ ) *If  $f \in L^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  and  $s > \frac{1}{2}$ , the mapping*

$$\varphi \mapsto \int_0^{2\pi} \langle f(t), \varphi(t) \rangle_{\mathbb{R}^d} dt$$

defines a linear form on  $H^s$ . Moreover,  $J^{-s} f \in L^2$  and, for any  $\varphi \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,

$$\langle J^{-s} f, J^s \varphi \rangle_{L^2} = \int_0^{2\pi} \langle f(t), \varphi(t) \rangle_{\mathbb{R}^d} dt.$$

**Lemma 2.5** ( $H^{1/2+\varepsilon} \hookrightarrow L^\infty$ ) *If  $f \in H^{1/2+\varepsilon}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ ,  $\varepsilon > 0$ , one obtains  $f \in L^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  satisfying  $\|f\|_{L^\infty} \leq C_\varepsilon \|f\|_{H^{1/2+\varepsilon}}$ .*

Be aware that  $H^{1/2} \not\hookrightarrow L^\infty$ , cf. [51, p. 273], but for  $s \in (0, \frac{1}{2})$  we obtain an embedding into  $L^p$  spaces as follows.

**Lemma 2.6** ( $H^s \hookrightarrow L^{2/(1-2s)-\varepsilon}$ ) *Let  $s \in (0, \frac{1}{2})$ ,  $f \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ , and  $p \in [2, \frac{2}{1-2s})$ . Then  $f \in L^p(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  and  $\|f\|_{L^p} \leq C_{s,p} \|f\|_{H^s}$ .*

The following statement gives a condition which permits to conclude that some  $H^s$  function, integrated with respect to some parameter, still belongs to  $H^s$ . Of course, there are many possible generalizations.

**Lemma 2.7** ( $H^s \hookrightarrow \int H^s$ ) *Let  $d_1, d_2 \in \mathbb{N}$ ,  $s \geq 0$ , and  $Y \subset \mathbb{R}^{d_1}$ ,  $Z \subset \mathbb{R}^{d_2}$  be (Lebesgue) measurable. Moreover, let  $f : \mathbb{R}/2\pi\mathbb{Z} \times Y \times Z \rightarrow \mathbb{R}$  and  $g : Y \times Z \rightarrow \mathbb{R}$  be measurable functions such that*

$$\begin{aligned} \text{for a. e. } (y, z) \in Y \times Z : & \quad f(\cdot, y, z) \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d), \quad \|f(\cdot, y, z)\|_{H^s} \leq g(y, z), \\ \text{for a. e. } y \in Y : & \quad g(y, \cdot) \in L^1(Z), \quad \operatorname{ess\,sup}_{\eta \in Y} \|g(\eta, \cdot)\|_{L^1} < \infty. \end{aligned}$$

Then we obtain  $F(\cdot, y) := \int_Z f(\cdot, y, z) \, dz \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  for a. e.  $y \in Y$  with

$$\|F(\cdot, y)\|_{H^s} \leq \operatorname{ess\,sup}_{\eta \in Y} \|g(\eta, \cdot)\|_{L^1} < \infty.$$

## 2.1 Derivation of the Euler-Lagrange equation

In the sequel we will denote by  $H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  those curves  $\gamma \in H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  whose curvature is *cube-integrable*, i. e.  $\|\dot{\gamma}\|_{L^3}^3 = \int_0^{2\pi} |\dot{\gamma}(t)|^3 \, dt < \infty$ . In the same way,  $H^{1,3}$  denotes an  $H^1$  function which possesses a cube-integrable first derivative.

Let  $\gamma \in H_{\text{ir}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  be a *critical point* of  $\tilde{E}^{(\alpha)} = \mathcal{L}^{\alpha-2} E^{(\alpha)}$ , i. e.

$$\delta \tilde{E}^{(\alpha)}(\gamma, h) = 0 \quad \text{for all } h \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d). \quad (2.2)$$

Since

$$\delta \tilde{E}^{(\alpha)}(\gamma, h) = (\alpha - 2) \mathcal{L}(\gamma)^{\alpha-3} E^{(\alpha)}(\gamma) \delta \mathcal{L}(\gamma, h) + \mathcal{L}(\gamma)^{\alpha-2} \delta E^{(\alpha)}(\gamma, h),$$

equation (2.2) is equivalent to

$$\delta E^{(\alpha)}(\gamma, h) = \ell^{(\alpha)}(\gamma) \left\langle \left( \frac{\dot{\gamma}}{|\dot{\gamma}|} \right)', h \right\rangle_{L^2}, \quad (2.3)$$

where  $\ell^{(\alpha)}(\gamma) := 2\pi \cdot \frac{\alpha-2}{\mathcal{L}(\gamma)} E^{(\alpha)}(\gamma)$  is finite for any  $\gamma \in H_{\text{ir}}^2$  by Theorem 1.6.

The ansatz via *Lagrange multipliers* also leads to (2.3) with the multiplier  $\ell^{(\alpha)}(\gamma)$ .

**Lemma 2.8** *Let  $\gamma \in H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  be a critical point of  $\tilde{E}^{(\alpha)}$ . Then  $h \mapsto Q^{(\alpha)}(\gamma, h)$  continuously extends to a (linear) functional on  $H^{1,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .*

**Proof.** Revisiting the proof of Lemma 1.17, the functional  $I^{(\alpha)}(\gamma, h) - \alpha Q^{(\alpha)}(\gamma, h)$ , originally defined on  $(\gamma, h) \in H_{\text{ia}}^2 \times H^2$ , continuously extends to  $H_{\text{ia}}^{2,3} \times H^{1,3}$ , for, applying (1.23) and (1.24), the first term on the right-hand side of (1.22) becomes

$$\begin{aligned} & \alpha \iint_{W_\varepsilon} \frac{1}{|w|^{\alpha-2}} \cdot \frac{X_\gamma}{w} \left[ \langle \dot{\gamma}(t), \dot{h}(t) \rangle - \left(\frac{\alpha}{2} + 1\right) \frac{\langle \Delta \gamma, \Delta h \rangle}{w^2} \right] \, dw \, dt \\ &= \alpha \iiint \iiint \iiint_{W_\varepsilon \times [0,1]^4} \frac{1}{|w|^{\alpha-2}} \cdot \left( 2u(1-u) \langle \ddot{\gamma}(t+uvw), \ddot{\gamma}(t+uw) \rangle_{\mathbb{R}^d} \right. \\ & \quad \left. - (1-u)(1-v) \langle \ddot{\gamma}(t+uw), \ddot{\gamma}(t+vw) \rangle_{\mathbb{R}^d} \right) \\ & \quad \cdot \left( \langle \dot{\gamma}(t), \dot{h}(t) \rangle - \left(\frac{\alpha}{2} + 1\right) \langle \dot{\gamma}(t+\sigma w), \dot{h}(t+\tau w) \rangle_{\mathbb{R}^d} \right) \, d\sigma \, d\tau \, du \, dv \, dw \, dt. \end{aligned}$$

For any  $(\sigma, \tau, u, v, w) \in [0, 1]^4 \times ([-\pi, \pi] \setminus \{0\})$  the integrand clearly belongs to  $L^1$ . Its  $L^1$  norm is bounded by  $3\alpha(2 + \frac{\alpha}{2}) |w|^{2-\alpha} \|\ddot{\gamma}\|_{L^3}^2 \|\dot{\gamma}\|_{L^\infty} \|\dot{h}\|_{L^3}$ . Now, by FUBINI's theorem, the integrand is  $L^1(W_0 \times [0, 1]^4)$ , and the continuity on  $h \in H^{1,3}$  is obvious. The remaining two terms may be treated in the same manner. (Here we do not need  $\ddot{\gamma}, \dot{h} \in L^3$ .) Now let  $h \in H^{1,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  be approximated by  $(h_n)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  with respect to the  $H^{1,3}$  norm. By the assumptions we deduce

$$I^{(\alpha)}(\gamma, h_n) \stackrel{\text{Thm. 1.21}}{=} \delta E^{(\alpha)}(\gamma, h_n) \stackrel{(2.3)}{=} \ell^{(\alpha)}(\gamma) \langle \dot{\gamma}, h_n \rangle_{L^2}. \quad (2.4)$$



The right-hand side is linear and continuous even on  $h \in L^2$ , so  $\lim_{n \rightarrow \infty} Q^{(\alpha)}(\gamma, h_n)$  exists and is continuous on  $H^{1,3}$ .  $\square$

**Remark 2.9** Since  $Q^{(2)}$  is a continuous functional on  $H^2 \times H^1$  according to Proposition 1.3, the information given in Lemma 2.8 is new only for  $\alpha > 2$ .

Before proceeding further, we briefly introduce the *orthogonal projection* onto  $\dot{\gamma}$  and  $\dot{\gamma}^\perp$  respectively, which is given by

$$S_{\dot{\gamma}} := \dot{\gamma} \otimes \dot{\gamma}, \quad P_{\dot{\gamma}^\perp} := \text{id}_{\mathbb{R}^d} - S_{\dot{\gamma}} = \text{Id}_{\mathbb{R}^d \times \mathbb{R}^d} - \dot{\gamma} \otimes \dot{\gamma}. \quad (2.5)$$

Here, for any two vectors  $u, v \in \mathbb{R}^d$ , the product  $u \otimes v$  is defined as the matrix  $u \cdot v^\top \in \mathbb{R}^{d \times d}$ , where  $v^\top$  denotes the transpose of  $v$ . The corresponding matrix-vector product for  $w \in \mathbb{R}^d$  equals  $(u \otimes v)w = u(v^\top w) = \langle v, w \rangle_{\mathbb{R}^d} u$ . Both projections are self-adjoint since  $\langle S_{\dot{\gamma}} u, v \rangle = \langle u, \dot{\gamma} \rangle \langle \dot{\gamma}, v \rangle = \langle u, S_{\dot{\gamma}} v \rangle$ .

**Lemma 2.10 (Orthogonal projection)** *Let  $s \geq 0$ ,  $\gamma \in H^{s+2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ , and  $g \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . Then  $S_{\dot{\gamma}}g, P_{\dot{\gamma}^\perp}g \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  and*

$$\|S_{\dot{\gamma}}g\|_{H^s} \leq C_{d,s} \|\dot{\gamma}\|_{H^{s+1}}^2 \|g\|_{H^s}, \quad \|P_{\dot{\gamma}^\perp}g\|_{H^s} \leq C_{d,s} \left(1 + \|\dot{\gamma}\|_{H^{s+1}}^2\right) \|g\|_{H^s}, \quad (2.6)$$

so  $S_{\dot{\gamma}}g, P_{\dot{\gamma}^\perp}g$  are (linear hence) continuous in  $g$ .

**Proof.**  $\|P_{\dot{\gamma}^\perp}g\|_{H^s} \leq \|g\|_{H^s} + \|S_{\dot{\gamma}}g\|_{H^s} \stackrel{\text{Cor. 2.3}}{\leq} \|g\|_{H^s} + C_{d,s} \|\dot{\gamma}\|_{H^{s+1}}^2 \|g\|_{H^s}$ .  $\square$

**Remark 2.11** In case  $s > \frac{1}{2}$  we only require  $\gamma \in H^{s+1}$  for the statement of Lemma 2.10 obtaining  $\|S_{\dot{\gamma}}g\|_{H^s} \leq C_{d,s} \|\dot{\gamma}\|_{H^s}^2 \|g\|_{H^s}$ , cf. Corollary 2.3 (i). — The projections are also continuous as operators  $S_\bullet, P_\bullet : H^{s+2} \rightarrow \text{Lin}(H^s)$ ,  $\gamma \mapsto S_\gamma$  or  $\gamma \mapsto P_{\dot{\gamma}^\perp}$ , with respect to the corresponding topology.

**Lemma 2.12** *Let  $\gamma \in H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  be a critical point of  $\tilde{E}^{(\alpha)}$  and  $g \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ . Then  $h := P_{\dot{\gamma}^\perp}g \in H^{1,3}$  and there is a (nonlinear) operator*

$$M^{(\alpha)} : H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \rightarrow L^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$$

satisfying

$$\ell^{(\alpha)}(\gamma) \langle \ddot{\gamma}, h \rangle_{L^2} - \alpha Q^{(\alpha)}(\gamma, h) = \int_0^{2\pi} \left\langle (M^{(\alpha)}\gamma)(t), h(t) \right\rangle_{\mathbb{R}^d} dt. \quad (2.7)$$

**Proof.** Obviously  $\dot{h} = (g - \langle g, \dot{\gamma} \rangle \dot{\gamma})' = \dot{g} - \langle \dot{g}, \dot{\gamma} \rangle \dot{\gamma} - \langle g, \ddot{\gamma} \rangle \dot{\gamma} - \langle g, \dot{\gamma} \rangle \ddot{\gamma}$  belongs to  $L^3$  since  $\dot{g}, \dot{\gamma} \in L^\infty$ ,  $\ddot{\gamma} \in L^3$ . Let  $\varphi \in L^1(W_0)$  be  $2\pi$ -periodic in the first argument and satisfy  $\varphi(t+w, -w) = \varphi(t, w)$  on  $W_0 = [0, 2\pi] \times [-\pi, \pi]$ . By  $X_\gamma(t+w, -w) = -X_\gamma(t, w)$ , the functions

$$\begin{aligned} \varphi_1(t, w) &:= \frac{1}{|w|^{\alpha-2}} \cdot \frac{X_\gamma(t, w)}{w} \quad \text{and} \\ \varphi_2(t, w) &:= \left[ \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} + \frac{1}{1-wX_\gamma} \right) + \frac{w}{|w|^{\alpha-2}} X_\gamma^3 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} \frac{1}{1-wX_\gamma} \right) + \right. \\ &\quad \left. + |w|^{4-\alpha} \frac{X_\gamma^4}{1-wX_\gamma} \Omega_\gamma^{(\alpha)} \right] \end{aligned}$$

are admissible in this sense. Using the periodicity of  $\gamma$  and  $\varphi(\cdot, w)$ , we obtain

$$\iint_{W_\varepsilon} \varphi(t, w) \cdot \frac{\langle \gamma(t+w) - \gamma(t), h(t+w) \rangle}{w^2} dw dt = - \iint_{W_\varepsilon} \varphi(\bar{t}, \bar{w}) \cdot \frac{\langle \gamma(\bar{t}+\bar{w}) - \gamma(\bar{t}), h(\bar{t}) \rangle}{\bar{w}^2} d\bar{w} d\bar{t}.$$

Together with  $\langle \gamma(t+w) - \gamma(t), h(t) \rangle_{\mathbb{R}^d} = w^2 \left\langle \int_0^1 (1-\varrho) \ddot{\gamma}(t+\varrho w) d\varrho, h(t) \right\rangle_{\mathbb{R}^d}$  on  $W_0$  which stems from  $\langle \dot{\gamma}, h \rangle \equiv 0$ , we arrive at

$$\iint_{W_\varepsilon} \varphi(t, w) \frac{\langle \Delta \gamma, \Delta h \rangle}{w^2} dw dt = -2 \iint_{W_\varepsilon} \varphi(t, w) \left\langle \int_0^1 (1-\varrho) \ddot{\gamma}(t+\varrho w) d\varrho, h(t) \right\rangle dw dt.$$

Transforming the right-hand side of (1.22) for  $\varphi = \varphi_1, \varphi_2$  and  $\varepsilon \searrow 0$  in this manner, deducing from  $\langle \dot{\gamma}, h \rangle \equiv 0$  that  $\langle \dot{\gamma}, \dot{h} \rangle \equiv -\langle \ddot{\gamma}, h \rangle$  and recalling (2.4), the left-hand side of (2.7) equals

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \left\{ \alpha \iint_{W_\varepsilon} \frac{1}{|w|^{\alpha-2}} \cdot \frac{X_\gamma}{w} \left\langle -\ddot{\gamma}(t) + (\alpha+2) \int_0^1 (1-\varrho) \ddot{\gamma}(t+\varrho w) d\varrho, h(t) \right\rangle dw dt \right. \\ & \quad - 2 \iint_{W_\varepsilon} \langle \ddot{\gamma}(t), h(t) \rangle \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \Omega_\gamma^{(\alpha)} dw dt \\ & \quad + 2\alpha \iint_{W_\varepsilon} \left\langle \int_0^1 (1-\varrho) \ddot{\gamma}(t+\varrho w) d\varrho, h(t) \right\rangle \left[ \frac{X_\gamma^2}{|w|^{\alpha-2}} \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} + \frac{1}{1-wX_\gamma} \right) \right. \\ & \quad \left. \left. + \frac{w}{|w|^{\alpha-2}} X_\gamma^3 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} \frac{1}{1-wX_\gamma} \right) + |w|^{4-\alpha} \frac{X_\gamma^4}{1-wX_\gamma} \Omega_\gamma^{(\alpha)} \right] dw dt \right\}. \end{aligned} \quad (2.8)$$

Now let  $(M^{(\alpha)\gamma})(t) :=$

$$\begin{aligned} & 2\pi \left\{ \alpha \int_{-\pi}^{\pi} \frac{1}{|w|^{\alpha-2}} \cdot \frac{X_\gamma}{w} \left[ -\ddot{\gamma}(t) + (\alpha+2) \int_0^1 (1-\varrho) \ddot{\gamma}(t+\varrho w) d\varrho \right] dw \right. \\ & \quad - 2 \int_{-\pi}^{\pi} \ddot{\gamma}(t) \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \Omega_\gamma^{(\alpha)} dw \\ & \quad + 2\alpha \int_{-\pi}^{\pi} \int_0^1 (1-\varrho) \ddot{\gamma}(t+\varrho w) d\varrho \left[ \frac{1}{|w|^{\alpha-2}} X_\gamma^2 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} + \frac{1}{1-wX_\gamma} \right) \right. \\ & \quad \left. \left. + \frac{w}{|w|^{\alpha-2}} X_\gamma^3 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} \frac{1}{1-wX_\gamma} \right) + |w|^{4-\alpha} \frac{X_\gamma^4}{1-wX_\gamma} \Omega_\gamma^{(\alpha)} \right] dw \right\}. \end{aligned} \quad (2.9)$$

Recalling that the functions  $X_\gamma$ ,  $\Omega_\gamma^{(\alpha)}$ , and  $\frac{1}{1-wX_\gamma}$  belong to  $L^\infty(W_0)$  by Corollary 1.5 and additionally using (1.23), we see that for each tuple  $(\varrho, u, v, w) \in [0, 1]^3 \times [-\pi, \pi] \setminus \{0\}$  the integrand on the right-hand side of (2.9) belongs to  $L^1(\mathbb{R}/2\pi\mathbb{Z})$  and is bounded by  $C_{\alpha, \gamma} |w|^{2-\alpha}$ . So, by FUBINI's theorem, the integrand is  $L^1(W_0 \times [0, 1]^3)$  which implies  $M^{(\alpha)\gamma} \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ . Finally, we conclude the proof by writing (2.8) as  $\int_0^{2\pi} \langle (M^{(\alpha)\gamma})(t), h(t) \rangle_{\mathbb{R}^d} dt$ .  $\square$

**Remark 2.13** Using Corollary 1.5 we can also show the continuity of  $\gamma \mapsto M^{(\alpha)\gamma}$ .

**Lemma 2.14 (Commutator)** Let  $\gamma \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  and  $\mu_{k, \ell} := -\sigma_{k+\ell, \ell}^{(\alpha-1)}$ . Then the linear operator  $N_\gamma^{(\alpha)} : g \mapsto \mathcal{M}_\mu(J^{\alpha-1} S_\gamma, J^{\alpha-2} g)$  from (2.1) satisfies

$$\begin{aligned} (i) \quad & \gamma \in H^\alpha, \quad g \in H^{\alpha-2}, \quad \varepsilon > 0 \implies J^{-\frac{1}{2}-\varepsilon} N_\gamma^{(\alpha)} g \in L^2, \\ (ii) \quad & \gamma \in H^{\alpha+\frac{1}{2}-\varepsilon}, \quad g \in H^{\alpha-2+\frac{1}{2}-\varepsilon}, \quad \varepsilon \leq \frac{1}{2} \implies J^{-2\varepsilon} N_\gamma^{(\alpha)} g \in L^2, \\ (iii) \quad & \gamma \in H^{\alpha+s}, \quad g \in H^{\alpha-2+s}, \quad s > \frac{1}{2} \implies N_\gamma^{(\alpha)} g \in H^s. \end{aligned}$$

Moreover,  $N_\gamma^{(\alpha)}$  is skew-adjoint, i. e.  $\langle N_\gamma^{(\alpha)} g, h \rangle_{L^2} = -\langle g, N_\gamma^{(\alpha)} h \rangle_{L^2}$  for  $h \in H^{\frac{1}{2}+\varepsilon}$ ,  $\varepsilon > 0$ . For  $g \in H^{\alpha-1}$  we obtain, using the commutator bracket,

$$N_\gamma^{(\alpha)} g = -[J^{\alpha-1}, S_\gamma] g = [J^{\alpha-1}, P_{\dot{\gamma}\pm}] g = J^{\alpha-1} P_{\dot{\gamma}\pm} g - P_{\dot{\gamma}\pm} J^{\alpha-1} g. \quad (2.10)$$

Proof. Statements (i) – (iii) follow from Theorem 2.1 (i), (iii).  $\square$

**Remark 2.15** As operator  $N^{(\alpha)} : H^{\alpha+s} \rightarrow \text{Lin}(H^{\alpha-2+s}, H^s)$ ,  $\gamma \mapsto N_\gamma^{(\alpha)}$ , the commutator is continuous with respect to the corresponding topology. The argument is parallel to that given in Remark 2.11.

Recalling (1.9), (1.12), the operator  $J^{\alpha-1} + \frac{1}{a^{(\alpha)}}L^{(\alpha)} - \frac{\ell^{(\alpha)}(\gamma)}{\alpha a^{(\alpha)}}\frac{d^2}{dt^2}$  linearly maps  $H^{s+2}$  to  $H^s$  such that

$$\left\| \left( J^{\alpha-1} + \frac{1}{a^{(\alpha)}}L^{(\alpha)} - \frac{\ell^{(\alpha)}(\gamma)}{\alpha a^{(\alpha)}}\frac{d^2}{dt^2} \right) \gamma \right\|_{H^s} \leq C \left( \alpha, E^{(\alpha)}(\gamma), \mathcal{L}(\gamma) \right) \|\gamma\|_{H^{s+2}}.$$

Now (2.6) yields

**Lemma 2.16** For any  $s \geq 0$ ,

$$\gamma \mapsto R^{(\alpha)}\gamma := P_{\dot{\gamma}^\perp} \left[ \left( J^{\alpha-1} + \frac{1}{a^{(\alpha)}}L^{(\alpha)} - \frac{\ell^{(\alpha)}(\gamma)}{\alpha a^{(\alpha)}}\frac{d^2}{dt^2} \right) \gamma \right]$$

defines a (nonlinear) operator  $H^{s+2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \rightarrow H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  satisfying

$$\|R^{(\alpha)}\gamma\|_{H^s} \leq C \left( 1 + \|\gamma\|_{H^{s+2}}^3 \right), \quad C = C \left( \alpha, d, s, E^{(\alpha)}(\gamma), \mathcal{L}(\gamma) \right).$$

**Remark 2.17** The operator  $\gamma \mapsto R^{(\alpha)}\gamma$  is continuous, cf. Remark 2.11. Recall the continuity of  $\mathcal{L}$  and  $E^{(\alpha)}$  (Theorem 1.6).

**Corollary 2.18 (Weak Euler-Lagrange equation for  $\tilde{E}^{(\alpha)}$ )** Let  $\gamma \in H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  be a critical point of  $\tilde{E}^{(\alpha)}$ . Then

$$\langle \ddot{\gamma}, J^{\alpha-1}g \rangle_{L^2} = \langle R^{(\alpha)}\gamma, g \rangle_{L^2} - \langle \ddot{\gamma}, N_\gamma^{(\alpha)}g \rangle_{L^2} + \frac{1}{\alpha a^{(\alpha)}} \int_0^{2\pi} \langle (P_{\dot{\gamma}^\perp} M^{(\alpha)}\gamma)(t), g(t) \rangle dt \quad (2.11)$$

holds for all  $g \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .

Proof. Let  $h := P_{\dot{\gamma}^\perp}g$  be approximated in  $H^{1,3}$  by  $(h_n)_{n \in \mathbb{N}} \subset C^\infty$ , so (1.9) gives

$$a^{(\alpha)} \langle J^2\gamma, J^{\alpha-1}h_n \rangle_{L^2} = Q^{(\alpha)}(\gamma, h_n) - \langle L^{(\alpha)}\gamma, h_n \rangle_{L^2}. \quad (2.12)$$

Since the right-hand side continuously extends to  $h \in H^{1,3}$  by Proposition 1.3 (ii) and Lemma 2.8, the same holds true for the left-hand side, so we may pass to the limit  $n \rightarrow \infty$ . Using  $J^2\gamma = \gamma - \ddot{\gamma}$  we obtain

$$\begin{aligned} \frac{1}{a^{(\alpha)}} \left( Q^{(\alpha)}(\gamma, h) - \langle L^{(\alpha)}\gamma, h \rangle_{L^2} \right) &= \langle J^2\gamma, J^{\alpha-1}h \rangle_{L^2} = \langle \gamma, J^{\alpha-1}h \rangle_{L^2} - \langle \ddot{\gamma}, J^{\alpha-1}h \rangle_{L^2} \\ &\stackrel{(2.10)}{=} \langle \gamma, J^{\alpha-1}P_{\dot{\gamma}^\perp}g \rangle_{L^2} - \langle \ddot{\gamma}, N_\gamma^{(\alpha)}g \rangle_{L^2} - \langle \ddot{\gamma}, P_{\dot{\gamma}^\perp}J^{\alpha-1}g \rangle_{L^2}. \end{aligned}$$

Since  $|\dot{\gamma}| \equiv 1$  implies  $P_{\dot{\gamma}^\perp}\ddot{\gamma} = \ddot{\gamma}$  and  $P_{\dot{\gamma}^\perp}$  is self-adjoint, we arrive at

$$\begin{aligned} \langle \ddot{\gamma}, J^{\alpha-1}g \rangle_{L^2} &= \langle \gamma, J^{\alpha-1}P_{\dot{\gamma}^\perp}g \rangle_{L^2} - \langle \ddot{\gamma}, N_\gamma^{(\alpha)}g \rangle_{L^2} + \frac{1}{a^{(\alpha)}} \left( \langle L^{(\alpha)}\gamma, P_{\dot{\gamma}^\perp}g \rangle_{L^2} - Q^{(\alpha)}(\gamma, P_{\dot{\gamma}^\perp}g) \right) \\ &\stackrel{(2.7)}{=} \langle J^{\alpha-1}\gamma, P_{\dot{\gamma}^\perp}g \rangle_{L^2} - \langle \ddot{\gamma}, N_\gamma^{(\alpha)}g \rangle_{L^2} + \frac{1}{a^{(\alpha)}} \langle L^{(\alpha)}\gamma, P_{\dot{\gamma}^\perp}g \rangle_{L^2} \\ &\quad + \frac{1}{\alpha a^{(\alpha)}} \int_0^{2\pi} \langle (M^{(\alpha)}\gamma)(t), (P_{\dot{\gamma}^\perp}g)(t) \rangle_{\mathbb{R}^d} dt - \frac{\ell^{(\alpha)}(\gamma)}{\alpha a^{(\alpha)}} \langle \ddot{\gamma}, P_{\dot{\gamma}^\perp}g \rangle_{L^2}. \end{aligned}$$

$\square$

**Remark 2.19** For any critical point  $\gamma \in H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  of  $\tilde{E}^{(\alpha)}$  the linear form  $h \mapsto \langle J^2\gamma, J^{\alpha-1}h \rangle_{L^2} = \langle J^{\alpha+1}\gamma, h \rangle_{L^2}$  extends to  $H^{1,3}$  as shown in (2.12). Using theory for fractional  $L^p$ -Sobolev spaces, cf. e. g. [52, Sect. 13.6], we obtain  $J^{\alpha+1}\gamma \in (H^{1,3})^* \cong H^{-1,3/2}$ , so  $\gamma \in H^{\alpha,3/2}$ . Unfortunately, we need more integrability, namely  $H_{\text{ia}}^{\alpha,2} \cap H^{2,3}$ , in order to establish the boot-strapping.

## 2.2 The Monster operator

**Lemma 2.20**  $\gamma \in H_{\text{ia}}^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d), \ddot{\gamma} \in L^6 \implies M^{(\alpha)}\gamma \in L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .

*Proof.* Denoting the integrands of the three terms on the right-hand side of (2.9) by  $m_i, i = 1, 2, 3$ , the function  $M^{(\alpha)}\gamma$  reads

$$(M^{(\alpha)}\gamma)(t) = 2\pi \sum_{j=1}^3 \int_{-\pi}^{\pi} \frac{m_j(\gamma; t, w)}{|w|^{\alpha-2}} dw. \quad (2.13)$$

Using (1.5) together with FUBINI's theorem and SCHWARZ' inequality, we arrive at  $\int_0^{2\pi} |m_i(\gamma; t, w)|^2 dt \leq C_{\alpha, \beta_\gamma} \max\left(1, \|\dot{\gamma}\|_{L^2}^{18}, \|\ddot{\gamma}\|_{L^6}^6\right)$  uniformly for a. e.  $w \in [-\pi, \pi]$ , i. e. the constants do not depend on  $w$ . The same bound applies to  $\|\dot{\gamma}\|_{L^6}^6$  by FUBINI's theorem and SCHWARZ' inequality. This is finite provided  $\ddot{\gamma} \in L^6$ .  $\square$

**Lemma 2.21**  $\gamma \in H_{\text{ia}}^{2+s}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d), s > \frac{1}{2} \implies M^{(\alpha)}\gamma \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ .

In the following proof we use the term “ $f(\cdot, y) \in H^s$  uniformly for a. e.  $y \in Y$ ” to abbreviate “ $f(\cdot, y) \in H^s$  for a. e.  $y \in Y$  and  $\text{ess sup}_{y \in Y} \|f(\cdot, y)\|_{H^s} < \infty$ ”.

*Proof.* Let  $s > \frac{1}{2}$  and  $\gamma \in H_{\text{ia}}^{2+s}$ . Corollary 2.3 implies

$$\begin{aligned} (t \mapsto f(t, u, v, w) := & 2(1-u) \langle \dot{\gamma}(t+uw) - \dot{\gamma}(t), \ddot{\gamma}(t+uw) \rangle \\ & - w \langle (1-u)\ddot{\gamma}(t+uw), (1-v)\ddot{\gamma}(t+vw) \rangle) \in H^s(\mathbb{R}/2\pi\mathbb{Z}) \end{aligned}$$

uniformly on  $(u, v, w) \in [0, 1]^2 \times [-\pi, \pi]$ , for

$$\|f(\cdot, u, v, w)\|_{H^s} \leq C_s \left( \|\dot{\gamma}\|_{H^s} \|\ddot{\gamma}\|_{H^s} + \|\ddot{\gamma}\|_{H^s}^2 \right).$$

Now Lemma 2.7 yields by (1.4), for  $w \in Y := [-\pi, \pi], (u, v) \in Z := [0, 1]^2$ ,

$$X_\gamma(\cdot, w) \in H^s \quad \text{uniformly for a. e. } w \in [-\pi, \pi]. \quad (2.14)$$

By Corollary 2.3 and Lemma 2.7 ( $w \in Y := [-\pi, \pi], \vartheta \in Z := [0, 1]$ ), the right-hand side of  $0 < \beta_\gamma \leq 1 - wX_\gamma(t, w) = \left| \frac{\Delta \gamma}{w} \right|^2 = \left| \int_0^1 \dot{\gamma}(t + \vartheta w) d\vartheta \right|^2$  belongs to  $H^{1+s}$  uniformly for a. e.  $w \in [-\pi, \pi]$ . Since there is a (unique)  $k \in \mathbb{N}$  such that  $s \leq k < 1 + s$ , the quotient rule for  $H^k$  functions gives

$$(1 - wX_\gamma(\cdot, w))^{-1} \in H^k \subset H^s \quad \text{uniformly for a. e. } w \in [-\pi, \pi], \quad (2.15)$$

which of course also holds for  $(1 - \mu wX_\gamma(\cdot, w))^{-1}$  uniformly for a. e.  $(\mu, w) \in [0, 1] \times [-\pi, \pi]$ . Since we may use the usual differentiation rules on  $H^k$ , we even obtain  $(1 - \mu)(1 - \mu wX_\gamma(t, w))^{-\alpha/2-2} \in H^k$  uniformly for a. e.  $w \in [-\pi, \pi], \mu \in [0, 1]$ , so by (1.5) and Lemma 2.7 ( $w \in Y := [-\pi, \pi], \mu \in Z := [0, 1]$ )

$$\Omega_\gamma^{(\alpha)}(\cdot, w) \in H^k \subset H^s \quad \text{uniformly for a. e. } w \in [-\pi, \pi]. \quad (2.16)$$

By (2.14), (2.15), (2.16), and Corollary 2.3, the functions

$$\begin{aligned} \tilde{m}_1(\gamma; \varrho, t, u, v, w) := & \alpha \left[ 2u(1-u) \langle \ddot{\gamma}(t+uvw), \ddot{\gamma}(t+uw) \rangle_{\mathbb{R}^d} \right. \\ & \left. - (1-u)(1-v) \langle \ddot{\gamma}(t+uw), \ddot{\gamma}(t+vw) \rangle_{\mathbb{R}^d} \right] \\ & \cdot [-\ddot{\gamma}(t) + (\alpha+2)(1-\varrho)\ddot{\gamma}(t+\varrho w)], \\ \tilde{m}_2(\gamma; t, w) := & -2\ddot{\gamma}(t)X_\gamma^2\Omega_\gamma^{(\alpha)}, \end{aligned}$$

$$\begin{aligned} \tilde{m}_3(\gamma; \varrho, t, w) := & 2\alpha(1 - \varrho)\ddot{\gamma}(t + \varrho w) \left[ X_\gamma^2 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} + \frac{1}{1 - wX_\gamma} \right) \right. \\ & \left. + wX_\gamma^3 \left( \Omega_\gamma^{(\alpha)} + \frac{\alpha}{2} \frac{1}{1 - wX_\gamma} \right) + \frac{w^2 X_\gamma^4}{1 - wX_\gamma} \Omega_\gamma^{(\alpha)} \right] \end{aligned}$$

belong to  $H^s$  with respect to the variable  $t$ . Their  $H^s$  norms are bounded uniformly for a. e.  $(\varrho, u, v, w) \in [0, 1]^3 \times [-\pi, \pi]$ . Using the notation from (2.13) at the beginning of the proof of Lemma 2.20 we obtain  $m_1(\gamma; t, w) = \iint_{[0,1]^3} \tilde{m}_1(\gamma; \varrho, t, u, v, w) d\varrho du dv$  and  $m_3(\gamma; t, w) = \int_0^1 \tilde{m}_3(\gamma; \varrho, t, w) d\varrho$ . Moreover,  $m_2(\gamma; \cdot, w) = \tilde{m}_2(\gamma; \cdot, w) \in H^s$  and, for  $i = 1, 3$ , Lemma 2.7 implies  $m_i(\gamma; \cdot, w) \in H^s$ , in all three cases uniformly for a. e.  $w \in [-\pi, \pi]$ . Finally, the right-hand side of (2.13) belongs to  $H^s$  by Lemma 2.7 ( $w \in Z := [-\pi, \pi]$ ) and  $\int_{-\pi}^\pi |w|^{2-\alpha} dw < \infty$ .  $\square$

If  $\gamma \in H^2$  the operator  $P_{\dot{\gamma}^\perp}$  maps  $L^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  to itself, which gives, together with (2.6) and Lemma 2.6

$$\text{Corollary 2.22} \quad \left\{ \begin{array}{ll} \text{(i)} & \text{If } \gamma \in H_{\text{ia}}^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \quad \text{then } P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma \in L^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d). \\ \text{(ii)} & \text{If } \gamma \in H_{\text{ia}}^{2+s}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \text{ and } s > \frac{1}{3} \text{ then } P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma \in L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d). \\ \text{(iii)} & \text{If } \gamma \in H_{\text{ia}}^{2+s}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \text{ and } s > \frac{1}{2} \text{ then } P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma \in H^s(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d). \end{array} \right.$$

### 2.3 Bootstrapping argument

**Theorem 2.23 (Regularity of  $\tilde{E}^{(\alpha)}$ -critical points)** Any critical point  $\gamma \in H_{\text{ia}}^\alpha(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$  of  $\tilde{E}^{(\alpha)}$  with cube-integrable curvature is  $C^\infty$ -smooth.

**Remark 2.24** If  $\alpha > 2 + \frac{1}{6}$  any  $H_{\text{ia}}^\alpha$  curve has already a cube-integral curvature by Lemma 2.6. (In fact, this is also true for  $\alpha = 2 + \frac{1}{6}$ .)

*Proof of Theorem 2.23.* Applying Lemma 2.14 ( $N_\gamma^{(\alpha)}$  skew-adjoint), Corollary 2.22 (i), and Lemma 2.4 to Equation (2.11) the term  $\langle \ddot{\gamma}, J^{\alpha-1} g \rangle_{L^2}$  equals

$$\left\langle R^{(\alpha)} \gamma, g \right\rangle_{L^2} + \left\langle J^{-\frac{1}{2}-\varepsilon} N_\gamma^{(\alpha)} \ddot{\gamma}, J^{\frac{1}{2}+\varepsilon} g \right\rangle_{L^2} + \frac{1}{\alpha a^{(\alpha)}} \left\langle J^{-\frac{1}{2}-\varepsilon} P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma, J^{\frac{1}{2}+\varepsilon} g \right\rangle_{L^2}$$

for any  $\varepsilon > 0$ . We test this identity using  $L^2$  basis elements  $g_{j,n} := \mathbf{e}_j \phi_n$ ,  $j \in \{1, \dots, d\}$ ,  $n \in \mathbb{Z}$ , where  $\mathbf{e}_j$  denotes the  $j$ -th unit vector in  $\mathbb{R}^d$ . Their  $m$ -th Fourier coefficient obviously amounts to  $\widehat{g_{j,n}}_m = \delta_{n,m} \mathbf{e}_j$ .

Dividing by  $(1 + m^2)^{\frac{1}{4} + \frac{1}{2}\varepsilon}$ , the Fourier coefficient  $\left( J^{(\alpha-2) + (\frac{1}{2}-\varepsilon)} \ddot{\gamma} \right)_m$  reads

$$\left( J^{-\frac{1}{2}-\varepsilon} R^{(\alpha)} \gamma \right)_m + \left( J^{-\frac{1}{2}-\varepsilon} N_\gamma^{(\alpha)} \ddot{\gamma} \right)_m + \frac{1}{\alpha a^{(\alpha)}} \left( J^{-\frac{1}{2}-\varepsilon} P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma \right)_m. \quad (2.17)$$

By Lemmata 2.16, 2.14 (i) and Corollary 2.22 (i) applied to Lemma 2.4, Equation (2.17) is  $\ell^2$  summable provided  $\varepsilon \in (0, \frac{1}{2}]$ . Multiplying by  $\phi_m$  and taking the sum over  $m \in \mathbb{Z}$ , we arrive at

$$J^{(\alpha-2) + (\frac{1}{2}-\varepsilon)} \ddot{\gamma} = J^{-\frac{1}{2}-\varepsilon} R^{(\alpha)} \gamma + J^{-\frac{1}{2}-\varepsilon} N_\gamma^{(\alpha)} \ddot{\gamma} + \frac{1}{\alpha a^{(\alpha)}} J^{-\frac{1}{2}-\varepsilon} P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma \in L^2. \quad (2.18)$$

This yields  $\gamma \in H^{\alpha + \frac{1}{2} - \varepsilon}$ . By Lemmata 2.16, 2.14 (ii), and Corollary 2.22 (ii) the right hand side of (2.18) is  $H^{\frac{1}{2} - \varepsilon}$  provided  $\varepsilon \in (0, \frac{1}{6})$ , so

$$J^{(\alpha-2) + (1-2\varepsilon)} \ddot{\gamma} = J^{-2\varepsilon} R^{(\alpha)} \gamma + J^{-2\varepsilon} N_\gamma^{(\alpha)} \ddot{\gamma} + \frac{1}{\alpha a^{(\alpha)}} J^{-2\varepsilon} P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma \in L^2. \quad (2.19)$$

This yields  $\gamma \in H^{\alpha+1-2\varepsilon}$ . By Lemmata 2.16, 2.14 (iii), and Corollary 2.22 (iii) the right hand side of (2.19) is  $H^1$  provided  $\varepsilon \in (0, \frac{1}{6})$ , so

$$J^{\alpha-1} \ddot{\gamma} = R^{(\alpha)} \gamma + N_\gamma^{(\alpha)} \ddot{\gamma} + \frac{1}{\alpha a^{(\alpha)}} P_{\dot{\gamma}^\perp} M^{(\alpha)} \gamma. \quad (2.20)$$

If now  $\gamma \in H^{\alpha+s}$  for any  $s > \frac{1}{2}$ , then by the same arguments the right-hand side of (2.20) belongs to  $H^s$ , such that  $\gamma \in H^{\alpha+s+1}$ . This gives the bootstrapping.  $\square$

## A Arc-length reparametrization preserves $H^2$ convergence

The appendix contains the proof of

**Theorem A.1** *The (rescaling and) reparametrizing operator*

$$\begin{aligned} \sim : H_{\text{ir}}^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d) &\longrightarrow H_{\text{ia}}^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d), \\ \gamma &\longmapsto \frac{\ell}{\mathcal{L}(\gamma)} \gamma \circ \psi_\gamma^{-1}, \end{aligned}$$

is continuous with respect to the  $H^2$  topology, where  $\psi_\gamma(t) := \frac{\ell}{\mathcal{L}(\gamma)} \mathcal{L}(\gamma|_{[0,t]})$ .

In the sequel we need the change-of-variables rule [41, p. 156 top] for Lebesgue measurable functions  $f$  and absolutely continuous monotone  $g : [a, b] \rightarrow \mathbb{R}$ . If one side of

$$\int_{g(a)}^{g(b)} f(t) dt = \int_a^b f(g(\tau)) \dot{g}(\tau) d\tau \quad (\text{A.1})$$

exists and is finite, then the same holds true for the other side, and the equation holds. By  $AC([a, b])$  we denote the set of absolutely continuous functions  $[a, b] \rightarrow \mathbb{R}$ . Recall  $AC([a, b]) = H^{1,1}(a, b)$ .

**Lemma A.2** *Let  $I = (a, b)$ ,  $f \in L^2(I)$ . Furthermore, let  $(u_k)_{k \in \mathbb{N}}$  be a sequence of absolutely continuous functions  $u_k : \bar{I} \rightarrow \bar{I}$  satisfying  $\dot{u}_k \geq c > 0$  for all  $k \in \mathbb{N}$  and (uniformly) converging to  $\text{id}_{\bar{I}}$  in  $C^0(\bar{I})$ . Then  $f \circ u_k \rightarrow f$  in  $L^2(I)$ .*

**Proof.** Let  $f$  be approximated by a sequence  $(f_j)_{j \in \mathbb{N}} \subset C^\infty$  with  $\|f_j - f\|_{L^2} \rightarrow 0$  as  $j \rightarrow \infty$ . Substituting  $t := u_k(\tau)$  we infer from (A.1)

$$\int_a^b |f(u_k(\tau)) - f_j(u_k(\tau))|^2 d\tau \leq \frac{1}{c} \int_{u_k(a)}^{u_k(b)} |f(t) - f_j(t)|^2 dt$$

which gives

$$\begin{aligned} \|f \circ u_k - f\|_{L^2} &\leq \|f \circ u_k - f_j \circ u_k\|_{L^2} + \|f_j \circ u_k - f_j\|_{L^2} + \|f_j - f\|_{L^2} \\ &\leq \left( \frac{1}{\sqrt{c}} + 1 \right) \|f_j - f\|_{L^2} + \sqrt{b-a} \|f_j \circ u_k - f_j\|_{L^\infty}. \end{aligned}$$

For fixed  $j$ , uniform convergence of  $(u_k)_{k \in \mathbb{N}}$  implies  $\|f_j \circ u_k - f_j\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow \infty$  which yields

$$\limsup_{k \rightarrow \infty} \|f \circ u_k - f\|_{L^2} \leq \left( \frac{1}{\sqrt{c}} + 1 \right) \|f_j - f\|_{L^2}.$$

Now we may pass to the limit  $j \rightarrow \infty$ , and the right-hand side vanishes by assumption.  $\square$

Next, we define  $H_c^2(0, \ell) := \left\{ \psi \in H^2(0, \ell) \mid \dot{\psi} > 0 \text{ on } [0, \ell], \psi(0) = 0, \psi(\ell) = \ell \right\}$ , the class of strictly increasing Sobolev functions keeping the endpoints fixed. Note that  $\inf_{[0, \ell]} \dot{\psi} > 0$  for each  $\psi \in H_c^2(0, \ell)$ .

**Theorem A.3** *The inversion operator  $^{-1} : H_c^2(0, \ell) \rightarrow H_c^2(0, \ell)$  is well-defined and continuous. Furthermore, for any  $\psi \in H_c^2(0, \ell)$ ,*

$$(\psi^{-1})^\cdot = \frac{1}{\dot{\psi} \circ \psi^{-1}}, \quad (\psi^{-1})^{\cdot\cdot} = -\frac{\ddot{\psi} \circ \psi^{-1}}{(\dot{\psi} \circ \psi^{-1})^3} \text{ a. e.} \quad (\text{A.2})$$

**Proof.** Changing  $\psi \in H^2(0, \ell)$  on a measure-zero set we obtain a  $C^1([0, \ell])$  function which possesses an inverse  $\phi := \psi^{-1} \in C^1([0, \ell])$  satisfying  $\dot{\phi} = \frac{1}{\dot{\psi} \circ \phi} \geq \frac{1}{\|\dot{\psi}\|_{L^\infty(0, \ell)}} > 0$ . Since  $\dot{\psi} \circ \phi \in H^1(0, \ell)$  with weak derivative  $(\dot{\psi} \circ \phi)^\cdot = (\ddot{\psi} \circ \phi) \dot{\phi}$  a. e. and  $r : [c, \infty) \rightarrow (0, \frac{1}{c}]$ ,  $t \mapsto t^{-1}$  is Lipschitz, we conclude  $\dot{\phi} = r \circ \dot{\psi} \circ \phi \in H^1(0, \ell)$  according to [53, Thm. 2.1.11]. This implies  $\phi \in H^2(0, \ell)$  with weak derivative  $\ddot{\phi} = -\frac{\ddot{\psi} \circ \phi}{(\dot{\psi} \circ \phi)^3}$  a. e. Now

we consider a sequence  $(\psi_n)_{n \in \mathbb{N} \cup \{0\}}$  in  $H_e^2(0, \ell)$ ,  $\psi_n \rightarrow \psi_0$  in  $H^2(0, \ell)$ . So we may assume  $\dot{\psi}_n \geq c$  for some  $c > 0$  and all  $n \in \mathbb{N} \cup \{0\}$ . We have to show  $\|\phi_k - \phi_0\|_{C^0([0, \ell])} + \|\dot{\phi}_k - \dot{\phi}_0\|_{C^0([0, \ell])} + \|\ddot{\phi}_k - \ddot{\phi}_0\|_{L^2(0, \ell)} \rightarrow 0$ . The essential part is to decompose  $\|\ddot{\psi}_k \circ \phi_k - \ddot{\psi}_0 \circ \phi_0\|_{L^2(0, \ell)} \leq \|\ddot{\psi}_k \circ \phi_k - \ddot{\psi}_0 \circ \phi_k\|_{L^2} + \|\ddot{\psi}_0 \circ \phi_k - \ddot{\psi}_0 \circ \phi_0\|_{L^2}$ . Applying (A.1) to  $t = \psi_k(\tau)$  we arrive at

$$\begin{aligned} \int_0^\ell \left| \ddot{\psi}_k(\phi_k(t)) - \ddot{\psi}_0(\phi_k(t)) \right|^2 dt &\leq \|\dot{\psi}_k\|_{C^0([0, \ell])} \|\ddot{\psi}_k - \ddot{\psi}_0\|_{L^2(0, \ell)}^2 \xrightarrow{k \rightarrow \infty} 0, \\ \int_0^\ell \left| \ddot{\psi}_0(\phi_k(t)) - \ddot{\psi}_0(\phi_0(t)) \right|^2 dt &\leq \|\dot{\psi}_k\|_{C^0([0, \ell])} \|\ddot{\psi}_0 \circ (\phi_0 \circ \psi_k) - \ddot{\psi}_0\|_{L^2(0, \ell)}^2. \end{aligned}$$

Due to the convergence  $\sup_{k \in \mathbb{N}} \|\dot{\psi}_k\|_{C^0([0, \ell])}$  is finite. The functions  $f := \ddot{\psi}_0$  and  $u_k := \phi_0 \circ \psi_k$  satisfy the conditions of Lemma A.2, which implies  $\|\ddot{\phi}_k - \ddot{\phi}_0\|_{L^2} \rightarrow 0$ .  $\square$

Applying (A.2) to the class of Sobolev functions reparametrizing a periodic function

$$H_p^2(0, \ell) := \left\{ \psi \in H^2(0, \ell) \mid \dot{\psi} > 0 \text{ on } [0, \ell], \psi(0) = 0, \psi(\ell) = \ell, \dot{\psi}(0) = \dot{\psi}(\ell) \right\} \subset H_e^2(0, \ell) \quad (\text{A.3})$$

we obtain

**Corollary A.4** *The restriction of the inversion operator  $^{-1}$  to  $H_p^2(0, \ell)$  is a continuous mapping onto  $H_p^2(0, \ell)$ .*

**Theorem A.5** *The composition operator*

$$\begin{aligned} H_p^2(0, \ell) \times H^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d) &\longrightarrow H^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d), \\ (\psi, h) &\longmapsto h \circ \psi, \end{aligned}$$

is well-defined and continuous. Moreover,

$$(h \circ \psi)' = (\dot{h} \circ \psi)\dot{\psi}, \quad (h \circ \psi)'' = (\ddot{h} \circ \psi)\dot{\psi}^2 + (\dot{h} \circ \psi)\ddot{\psi} \text{ a. e.} \quad (\text{A.4})$$

**Proof.** Since  $H^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d) \subset H^2((0, \ell), \mathbb{R}^d)$  we may first interpret  $h \circ \psi$  as a function from  $(0, \ell)$  to  $\mathbb{R}^d$ , which, by embeddings  $H_p^2(0, \ell) \times H^2((0, \ell), \mathbb{R}^d) \hookrightarrow C^1([0, \ell]) \times C^1([0, \ell], \mathbb{R}^d)$ , belongs to  $C^1([0, \ell], \mathbb{R}^d)$  and satisfies the first equation in (A.4).

Furthermore, since  $\psi$  is a  $C^1$ -diffeomorphism,  $\dot{h} \circ \psi \in H^1((0, \ell), \mathbb{R}^d)$  and  $(\dot{h} \circ \psi)' = (\ddot{h} \circ \psi)\dot{\psi}$ . From  $\dot{\psi} \in H^1$  we deduce  $(\dot{h} \circ \psi)\dot{\psi} \in H^1((0, \ell), \mathbb{R}^d)$  and the second equation in (A.4) follows.

Now  $(h \circ \psi)(0) = h(0) = h(\ell) = (h \circ \psi)(\ell)$  and  $(h \circ \psi)'(0) = \dot{h}(\psi(0))\dot{\psi}(0) = \dot{h}(0)\dot{\psi}(0) = (h \circ \psi)'(\ell)$  imply that the periodic extension of  $h \circ \psi$  to  $\mathbb{R}$  belongs to  $H^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d)$ . Using the technique from the preceding proof of Theorem A.3, we may show  $H^2$  convergence for any sequence  $(h_k, \psi_k)_{k \in \mathbb{N} \cup \{0\}}$  in  $H_p^2(0, \ell) \times H^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d)$ ,  $(h_k, \psi_k) \rightarrow (h_0, \psi_0)$ .  $\square$

**Proposition A.6** *The reparametrization operator*

$$\begin{aligned} \Psi : H_{\text{ir}}^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d) &\longrightarrow H_p^2(0, \ell), \\ \gamma &\longmapsto \left( t \mapsto \frac{\ell}{\mathcal{L}(\gamma)} \int_0^t |\dot{\gamma}(\tau)| d\tau \right), \end{aligned}$$

is continuous.

**Proof.** Let  $(\gamma_k)_{k \in \mathbb{N} \cup \{0\}}$  be a sequence of  $H_{\text{ir}}^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d)$  curves,  $\gamma_k \rightarrow \gamma_0$  in  $H^2$ . Due to the convergence there is some  $c > 0$  such that  $|\dot{\gamma}_k| \geq c > 0$  for all  $k \in \mathbb{N} \cup \{0\}$ . The functions  $\psi_k := \Psi\gamma_k$  are clearly  $C^1$  with  $\dot{\psi}_k(t) = \frac{\ell}{\mathcal{L}(\gamma_k)} |\dot{\gamma}_k(t)|$ , which is absolutely continuous by [53, Thm. 2.1.11] since  $r : \mathbb{R}^d \rightarrow [0, \infty)$ ,  $x \mapsto |x|$ , is Lipschitz, and we obtain  $\ddot{\psi}_k(t) = \frac{\ell}{\mathcal{L}(\gamma_k)} \cdot \frac{\langle \dot{\gamma}_k(t), \dot{\gamma}_k(t) \rangle}{|\dot{\gamma}_k(t)|}$ . Thus  $\psi_k \in H^2(0, \ell)$ , even  $\psi_k \in H_p^2(0, \ell)$ , and  $\|\psi_k - \psi_0\|_{H^2} \rightarrow 0$ .  $\square$

By Theorem A.3 the sequence of inverses  $\phi_k := \psi_k^{-1}$  also converges in  $H^2$ . Now we have collected all tools required for the

**Proof of Theorem A.1.** By Proposition A.6, the mapping  $\gamma \mapsto \Psi\gamma \in H_p^2(0, \ell)$  is continuous, hence, by Corollary A.4, the same holds true for  $\gamma \mapsto (\Psi\gamma)^{-1} \in H_p^2(0, \ell)$ . Theorem A.5 yields the continuity of  $\gamma \circ (\Psi\gamma)^{-1} \in H^2(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^d)$ . Of course,  $\gamma \circ (\Psi\gamma)^{-1}$  is injective and parametrized by constant velocity, for

$$\left| \left( \gamma \circ (\Psi\gamma)^{-1} \right) \cdot \right| \stackrel{(A.4)}{=} \left| \left( \dot{\gamma} \circ (\Psi\gamma)^{-1} \right) \left( (\Psi\gamma)^{-1} \right) \cdot \right| \stackrel{(A.2)}{=} \left| \frac{\dot{\gamma} \circ (\Psi\gamma)^{-1}}{(\Psi\gamma)' \circ (\Psi\gamma)^{-1}} \right| = \frac{\mathcal{L}(\gamma)}{\ell}.$$

Now the continuity of the length functional implies the desired.  $\square$

**Acknowledgements** This paper is based on my PhD thesis [40] at RWTH Aachen University. I would like to thank my advisor Prof. Dr. Heiko von der Mosel for his support. For many fruitful conversations, I am also indebted to Dr. Mads Kyed, Dr. Henryk Gerlach, and to Dr. Simon Blatt who suggested a shorter argument for Lemma A.2.

During a two-month research stay, partially supported by a *Doktorandenstipendium* of the German Academic Exchange Service (DAAD) and a grant of the German Research Foundation (DFG), I enjoyed the hospitality of the Laboratory for Computation and Visualization in Mathematics and Mechanics (LCVM<sup>2</sup>) at EPF Lausanne.

I would also like to express my gratitude to the referees for their valuable comments and for pointing out a mistake in the proof of Lemma A.2.

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